ON A THEOREM OF HUNEKE CONCERNING MULTIPLICITIES
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ABSTRACT. We consider a theorem by Huneke on multiplicities, and show that the extension surmised by Huneke is a stronger form of the Syzygy Conjecture.

Craig Huneke has proved the following result.

**Theorem 1** [9]. Let \((A, m)\) be a complete local ring containing a field, let \(n\) be a positive integer and let \(e(A)\) be the multiplicity of \(A\). Suppose that \(A\) satisfies Serre’s condition \(S_n\) and that \(e(A) \leq n\); if \(n = 1\) suppose further that \(A\) is equidimensional. Then \(A\) is Cohen-Macaulay.

All rings considered in this note are commutative, Noetherian and have an identity element, and all modules are finitely generated. For a discussion of the theory of (complete) local rings and of multiplicities, see [11, 12, 13, 14]. A finitely generated module \(M\) over a Noetherian ring \(A\) is said to satisfy \(S_n\) if

\[
\text{depth } M_P \geq \inf(n, \text{height } P) \quad \text{for all } P \in \text{Spec } A.
\]

As remarked on [11, p. 125], the condition \(S_1\) is equivalent to \(A\) having no embedded associated primes. We use the term ‘equidimensional’ in the sense of Grothendieck’s E.G.A., i.e. a local ring is said to be equidimensional when all its minimal primes have the same co-height. Recall from [7, (5.10.9)] that a catenary (and so a complete) local ring which satisfies \(S_2\) is equidimensional. A recent handy reference for much of the material in this note is the monograph [5], where a special case of Huneke’s result is discussed on pp. 71–72 and on p. 75. As the latter shows, and as has been remarked by Huneke himself, in the case \(n = 1\) it is essential in Theorem 1 to add the hypothesis of equidimensionality.

From his theorem Huneke deduces results by Nagata [12, (40.6)] and by Ikeda, in the case where the rings involved contain a field. (As regards the former result, see the interesting aside on [6, p. 41].) Since these are known to hold without this restriction, Huneke surmises that the following is probably true:

**Theorem 1** is valid without the assumption that \(A\) contain a field.

Now recall the Syzygy Conjecture [4, p. 143] for regular local rings, which runs as follows:

\[
\text{if } R \text{ is a regular local ring, and if } K \text{ is the } \\
\text{jth syzygy of an } R\text{-module with rank } K < j, \\
\text{then } K \text{ is a free } R\text{-module.}
\]
For a discussion of the links between jth syzygies and the condition $S_j$, and for a proof that (2) holds when $R$ contains a field, see [5, Chapter 3] and especially [loc. cit., Theorem 3.8 and Corollary 3.16]. It is also known that (2) holds when \( \dim R \leq 4 \) [4, p. 147].

Our final conjecture is the following:

\[
\text{Let } (R, m) \text{ be a regular local ring, let } (B, m') \text{ be a local ring which dominates [12, p. 14]} \text{ } R \text{ and suppose that } B \text{ is a finitely generated } R\text{-module. Let } n \text{ be a positive integer.}
\]

\[
\text{If } B \text{ satisfies } S_n \text{ and if the multiplicity } e_R(B) \leq n \text{ (where, if } n = 1, B \text{ is also supposed to be equidimensional), then } B \text{ is Cohen-Macaulay.}
\]

**Remarks.** Note that $B$ is catenary, being the homomorphic image of a polynomial ring over the regular ring $R$; note also that if $m \geq n$ then $S_m \Rightarrow S_n$. Hence, by [7, (5.10.9)], whatever the value of $n$ in (3), $B$ is equidimensional. Since $B$ is an integral extension of $R$, any two primes in $B$ having the same restriction to $R$ must have the same co-height in $B$ (see [10, Theorem 47]), and hence the same height in $B$. It follows from [7, (5.7.11)] that $B$ satisfies $S_n$ as a $B$-module if and only if it satisfies $S_n$ as an $R$-module.

**Theorem 2.** Conjectures (2) and (3) are equivalent, and are implied by Conjecture (1).

**Proof.** First suppose that (3) holds, and let $R$ and $K$ be as in the statement of (2). Consider the ring $B = R \times K$ formed by Nagata’s process of idealization [12, p. 2]. Then $B$ is a local ring which dominates $R$ and $B$ is a finitely generated $R$-module. Since $B = R \oplus K$ as an $R$-module, rank$_R B \leq j$. But as remarked in [9] (and as can easily be deduced from [12, §23])

\[
e_R(B) = \text{rank}_R B,
\]

where $e_R(B)$ is the multiplicity of $B$ as an $R$-module. By Auslander and Bridger’s theorem [1, (4.25)] (see [5, Theorem 3.8]), $K$ satisfies $S_j$. Now as $R$-modules, $B_P = R_P \oplus K_P$ for all $P \in \text{Spec } R$. By the definition of depth via the Ext-functor say (e.g. see [5, p. 3]), for all $P \in \text{Spec } R$,

\[
\text{depth}_R B_P = \min\{\text{depth}_R R_P, \text{depth}_R K_P\} = \min\{j, \text{height } P\},
\]

since $R$ is regular. Hence $B$ satisfies $S_j$ as an $R$-module, and so as a $B$-module, and $e_R(B) \leq j$. By (3) therefore, $B$ is Cohen-Macaulay. Thus $B$ is a free module, by [12, (25.16)], and therefore $K$ is also free, by [2, p. 147, Ex. 4] (or [5, p. 25, Ex. 10]).

Conversely, suppose that (2) holds, and let $R$ and $B$ be as in (3). Now $B$ satisfies $S_n$ as an $R$-module, and by the remarks above, rank$_R B \leq n$. Let $C = B/R$ and consider the exact sequence

\[
0 \to R \to B \to C \to 0.
\]

Then rank$_R C \leq n - 1$.

Fix $P \in \text{Spec } R$. Then we have the exact sequence

\[
0 \to R_P \to B_P \to C_P \to 0.
\]
First suppose that height \( P \leq n \). Then \( B_P \) is a Cohen-Macaulay \( R_P \)-module, and \( B_P \) is a semi-local integral over-ring of \( R_P \). It easily follows that \( B_P \) is itself a Cohen-Macaulay ring. Hence as before, by [12, (25.16)] and [2, p. 147, Ex. 4], \( C_P \) is a free \( R_P \)-module and so, by the definition of depth via the Ext-functor (say) and by the regularity of \( R \), \( \text{depth}_R C_P \) equals height \( P \).

On the other hand, suppose that height \( P > n \). It follows from the ‘Depth Lemma’ [5, p. 13], since \( R \) is regular, that \( \text{depth}_R C_P \geq \text{depth}_R B_P \) or \( \text{depth}_R C_P = \text{height} \ P - 1 \). Whatever the case therefore, \( \text{depth}_R C_P \geq n \) since \( B \) satisfies \( S_n \) as an \( R \)-module. Hence \( C \) also satisfies \( S_n \) as an \( R \)-module, and \( \text{rank}_R C \leq n - 1 \). By (2) therefore, \( C \) is free. Hence \( B \) is free, and so is Cohen-Macaulay by [12, (25.16)] again.

Finally consider Conjecture (1). Assume that it is valid. Then Conjecture (3) holds in the case where \( R \) is a complete regular local ring, for it easily follows that in this case \( B \) is also complete in the \( m' \)-adic (as well as in the \( m \)-adic) topology; by [14, Corollary 1, p. 299] and [12, (23.6)] \( e_R(B) \geq e(B) \). By Theorem 2 (and its proof), Conjecture (2) then holds in the case where \( R \) is complete. But by the faithful flatness of the completion functor, this is no restriction on Conjecture (2), which therefore holds as stated.

**COROLLARY 1.** Conjecture (3) holds when \( \dim R \) (= \( \dim B \)) \( \leq 4 \).

**PROOF.** See the remarks after the statement of Conjectures (2) and (3).

If one wishes to dispense with the restriction that \( R \) be a regular ring in Conjecture (3) and merely assume that \( B \) is a complete local ring, then it easily follows (as on [3, p. 59], say) that \( B \) is an equidimensional catenary ring which is a finitely generated torsion-free module over a complete local ring \( S \) which is either regular or a hypersurface (i.e. a regular local ring modulo a principal ideal); moreover, \( S \) contains a system of parameters for \( B \). It is not clear (to me) what else can usefully be said about this more general situation. The example given on pp. 27–28 of [8] is of interest in this regard, as it shows that the analogue of [12, (25.16)] for (complete) hypersurfaces (rather than regular local rings) is no longer valid.

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**REFERENCES**