PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 1, January 1987

ON A THEOREM OF HUNEKE CONCERNING MULTIPLICITIES

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ABSTRACT. We consider a theorem by Huneke on multiplicities, and show that the extension surmised by Huneke is a stronger form of the Syzygy Conjecture.

Craig Huneke has proved the following result.

THEOREM 1 [**9**]. *Let $(A, \underline{m})$ be a complete local ring containing a field, let n be a positive integer and let $e(A)$ be the multiplicity of A. Suppose that A satisfies Serre's condition S_n and that $e(A) \leq n$; if $n = 1$ suppose further that A is equidimensional. Then A is Cohen-Macaulay.*

All rings considered in this note are commutative, Noetherian and have an identity element, and all modules are finitely generated. For a discussion of the theory of (complete) local rings and of multiplicities, see [**11**, **12**, **13**, **14**]. A finitely generated module M over a Noetherian ring A is said to satisfy S_n if

$$\operatorname{depth} M_P \geq \inf(n, \operatorname{height} P) \quad \text{for all } P \in \operatorname{Spec} A.$$

As remarked on [**11**, p. 125], the condition S_1 is equivalent to A having no embedded associated primes. We use the term 'equidimensional' in the sense of Grothendieck's E.G.A., i.e. a local ring is said to be *equidimensional* when all its minimal primes have the same co-height. Recall from [**7**, (5.10.9)] that a catenary (and so a complete) local ring which satisfies S_2 is equidimensional. A recent handy reference for much of the material in this note is the monograph [**5**], where a special case of Huneke's result is discussed on pp. 71–72 and on p. 75. As the latter shows, and as has been remarked by Huneke himself, in the case $n = 1$ it is essential in Theorem 1 to add the hypothesis of equidimensionality.

From his theorem Huneke deduces results by Nagata [**12**, (40.6)] and by Ikeda, in the case where the rings involved contain a field. (As regards the former result, see the interesting aside on [**6**, p. 41].) Since these are known to hold without this restriction, Huneke surmises that the following is probably true:

(1) *Theorem 1 is valid without the assumption that A contain a field.*

Now recall the Syzygy Conjecture [**4**, p. 143] for regular local rings, which runs as follows:

(2) *if R is a regular local ring, and if K is the jth syzygy of an R-module with* $\operatorname{rank} K < j$, *then K is a free R-module.*

Received by the editors November 4, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 13H15.
Key words and phrases. Multiplicity, local ring, syzygy.

For a discussion of the links between jth syzygies and the condition S_j, and for a proof that (2) holds when R contains a field, see [**5**, Chapter 3] and especially [loc. cit., Theorem 3.8 and Corollary 3.16]. It is also known that (2) holds when $\dim R \leq 4$ [**4**, p. 147].

Our final conjecture is the following:

(3) *Let $(R, \underline{m})$ be a regular local ring, let $(B, \underline{m}')$ be a local ring which dominates* [**12**, *p.* 14] *R and suppose that B is a finitely generated R-module. Let n be a positive integer. If B satisfies S_n and if the multiplicity $e_R(B) \leq n$ (where, if $n = 1$, B is also supposed to be equidimensional), then B is Cohen-Macaulay.*

REMARKS. Note that B is catenary, being the homomorphic image of a polynomial ring over the regular ring R; note also that if $m \geq n$ then $S_m \Rightarrow S_n$. Hence, by [**7**, (5.10.9)], whatever the value of n in (3), B is equidimensional. Since B is an integral extension of R, any two primes in B having the same restriction to R must have the same co-height in B (see [**10**, Theorem 47]), and hence the same height in B. It follows from [**7**, (5.7.11)] that B satisfies S_n as a B-module if and only if it satisfies S_n as an R-module.

THEOREM 2. *Conjectures* (2) *and* (3) *are equivalent, and are implied by Conjecture* (1).

PROOF. First suppose that (3) holds, and let R and K be as in the statement of (2). Consider the ring $B = R \ltimes K$ formed by Nagata's process of idealization [**12**, p. 2]. Then B is a local ring which dominates R and B is a finitely generated R-module. Since $B = R \oplus K$ as an R-module, $\operatorname{rank}_R B \leq j$. But as remarked in [**9**] (and as can easily be deduced from [**12**, §23])

$$e_R(B) = \operatorname{rank}_R B,$$

where $e_R(B)$ is the multiplicity of B as an R-module. By Auslander and Bridger's theorem [**1**, (4.25)] (see [**5**, Theorem 3.8]), K satisfies S_j. Now as R-modules, $B_P = R_P \oplus K_P$ for all $P \in \operatorname{Spec} R$. By the definition of depth via the Ext-functor say (e.g. see [**5**, p. 3]), for all $P \in \operatorname{Spec} R$,

$$\operatorname{depth}_R B_P = \min\{\operatorname{depth}_R R_P, \operatorname{depth}_R K_P\} = \min\{j, \operatorname{height} P\},$$

since R is regular. Hence B satisfies S_j as an R-module, and so as a B-module, and $e_R(B) \leq j$. By (3) therefore, B is Cohen-Macaulay. Thus B is a free module, by [**12**, (25.16)], and therefore K is also free, by [**2**, p. 147, Ex. 4] (or [**5**, p. 25, Ex. 10]).

Conversely, suppose that (2) holds, and let R and B be as in (3). Now B satisfies S_n as an R-module, and by the remarks above, $\operatorname{rank}_R B \leq n$. Let $C = B/R$ and consider the exact sequence

$$0 \to R \to B \to C \to 0.$$

Then $\operatorname{rank}_R C \leq n - 1$.

Fix $P \in \operatorname{Spec} R$. Then we have the exact sequence

$$0 \to R_P \to B_P \to C_P \to 0.$$

First suppose that height $P \leq n$. Then B_P is a Cohen-Macaulay R_P-module, and B_P is a semi-local integral over-ring of R_P. It easily follows that B_P is itself a Cohen-Macaulay ring. Hence as before, by [**12**, (25.16)] and [**2**, p. 147, Ex. 4], C_P is a free R_P-module and so, by the definition of depth via the Ext-functor (say) and by the regularity of R, $\operatorname{depth}_R C_P$ equals height P.

On the other hand, suppose that height $P > n$. It follows from the 'Depth Lemma' [**5**, p. 13], since R is regular, that $\operatorname{depth}_R C_P \geq \operatorname{depth}_R B_P$ or $\operatorname{depth}_R C_P =$ height $P - 1$. Whatever the case therefore, $\operatorname{depth}_R C_P \geq n$ since B satisfies S_n as an R-module. Hence C also satisfies S_n as an R-module, and $\operatorname{rank}_R C \leq n - 1$. By (2) therefore, C is free. Hence B is free, and so is Cohen-Macaulay by [**12**, (25.16)] again.

Finally consider Conjecture (1). Assume that it is valid. Then Conjecture (3) holds in the case where R is a complete regular local ring, for it easily follows that in this case B is also complete in the $\underline{m}'$-adic (as well as the $\underline{m}$-adic) topology; by [**14**, Corollary 1, p. 299] and [**12**, (23.6)] $e_R(B) \geq e(B)$. By Theorem 2 (and its proof), Conjecture (2) then holds in the case where R is complete. But by the faithful flatness of the completion functor, this is no restriction on Conjecture (2), which therefore holds as stated.

COROLLARY 1. *Conjecture* (3) *holds when* $\dim R$ $(= \dim B) \leq 4$.

PROOF. See the remarks after the statement of Conjectures (2) and (3).

If one wishes to dispense with the restriction that R be a regular ring in Conjecture (3) and merely assume that B is a complete local ring, then it easily follows (as on [**3**, p. 59], say) that B is an equidimensional catenary ring which is a finitely generated torsion-free module over a complete local ring S which is either regular or a hypersurface (i.e. a regular local ring modulo a principal ideal); moreover, S contains a system of parameters for B. It is not clear (to me) what else can usefully be said about this more general situation. The example given on pp. 27–28 of [**8**] is of interest in this regard, as it shows that the analogue of [**12**, (25.16)] for (complete) hypersurfaces (rather than regular local rings) is no longer valid.

NOTE ADDED IN PROOF. I would like to thank S. Ikeda for pointing out an error in the original version of this paper, and for very valuable discussions.

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