AN UPPER BOUND FOR THE PERMANENT OF A 3-DIMENSIONAL (0,1)-MATRIX

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ABSTRACT. Let \( A = (a_{ijk}) \) be a 3-dimensional matrix of order \( n \). The permanent of \( A \) is defined by

\[
\text{per} A = \sum_{\sigma, \tau \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)\tau(i)},
\]

where \( S_n \) is the symmetric group on \( \{1, 2, \ldots, n\} \). Suppose that \( A \) is a (0,1)-matrix and that \( r_i = \sum_{j,k=1}^{n} a_{ijk} \) for \( i = 1, 2, \ldots, n \). In this paper it is shown that \( \text{per} A \leq \prod_{i=1}^{n} r_i^{1/r_i} \). A similar bound is then obtained for a second function, the 2-permanent of a 3-dimensional matrix, that is another analogue of the permanent of an ordinary (2-dimensional) matrix.

1. Introduction. It was conjectured by Minc [3] and proved by Brègman [1] that if \( A \) is a (0,1)-matrix of order \( n \) with row sums \( r_1, r_2, \ldots, r_n \), then

\[
\text{per} A \leq \prod_{i=1}^{n} r_i^{1/r_i},
\]

where by definition \( 0^{1/0} = 0 \). In this paper it is shown that an analogous result holds for 3-dimensional matrices. The permanent of a 3-dimensional matrix \( A = (a_{ijk}) \) of order \( n \) is defined by

\[
\text{(1)} \quad \text{per} A = \sum_{\sigma, \tau \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)\tau(i)}
\]

where \( S_n \) is the symmetric group on \( \{1, 2, \ldots, n\} \). We now state our principal result.

THEOREM 1. Let \( A = (a_{ijk}) \) be a 3-dimensional (0,1)-matrix of order \( n \), and let \( r_i = \sum_{j,k=1}^{n} a_{ijk} \) for \( i = 1, 2, \ldots, n \). Then

\[
\text{per} A \leq \prod_{i=1}^{n} r_i^{1/r_i}.
\]

The proof of Theorem 1, presented in §2, is an adaptation of Schrijver's proof [4] of the Minc-Brègman bound, with the extension to 3-dimensional matrices requiring a number of additional lemmas which may be of some independent interest.

In equation (1) and Theorem 1 the planes of \( A \) play the role of the rows and columns of an ordinary (2-dimensional) matrix, where a plane of a 3-dimensional
matrix \( A = (a_{ij}) \) is a submatrix obtained by fixing one of the indices \( i, j, k \) and letting the other two vary. We could also have taken the analogue of rows and columns to be the lines of \( A \), obtained by fixing two of these indices. In §3 we give a second definition of the permanent of a 3-dimensional matrix, called the 2-permanent, based on this observation. Using a result from [2] which relates these two types of permanents we obtain an upper bound for the 2-permanent of a 3-dimensional \((0, 1)\)-matrix similar to the one given in Theorem 1, with plane sums replaced by line sums.

2. Proof of the principal result. We write \( N, Z, \) and \( R^+ \) to denote, respectively, the set of natural numbers, the set of integers, and the set of positive real numbers. Also, for any real number \( x \), we use the notation \( \lfloor x \rfloor = \max \{ k \in Z : k \leq x \} \) and \( \lceil x \rceil = \min \{ k \in Z : x < k \} \). The first lemma appears in [4].

**Lemma 1.** If \( t = \sum_{i=1}^{n} t_i, t_i \in R^+, \) then \( t^t \leq n^t \prod_{i=1}^{n} t_i^t \).

**Lemma 2.** For all \( n \in N, n^{1/n}(n + 2)^{1/(n+2)} \leq (n + 1)^{2/(n+1)} \).

**Proof.** It is well known that if \( x_1, x_2, \ldots, x_k \in R^+ \), then \( \prod_{i=1}^{k} x_i \leq \overline{x}^k \) where \( \overline{x} = \left( \sum_{i=1}^{k} x_i \right) / k \). Let \( x_i = i \) for \( i = 1, 2, \ldots, n \), \( x_i = i - n \) for \( i = n + 1, n + 2, \ldots, 2n \), and \( x_i = n + 2 \) for \( i = 2n + 1, 2n + 2, \ldots, n^2 + 3n \). Then

\[
\prod_{i=1}^{n^2+3n} x_i = n!(n + 2)^{n^2+n}
\]

and

\[
\sum_{i=1}^{n^2+3n} x_i = (n^2 + 3n)(n + 1).
\]

Hence the inequality above becomes \( n!(n + 2)^{n^2+n} \leq (n + 1)^{n^2+3n} \), which is equivalent to the desired inequality.

**Lemma 3.** Suppose that a function \( f : N \to R^+ \) has the property that

\[
f(x)f(x + 2) \leq (f(x + 1))^2
\]

for all \( x \in N \), and that \( x_1, x_2, \ldots, x_k \in N \). Let \( \sum_{i=1}^{k} x_i = k\overline{x} = k\lfloor \overline{x} \rfloor + k_2 \lceil \overline{x} \rceil \), where \( k_1, k_2 \in Z, k_1, k_2 \geq 0, \) and \( k_1 + k_2 = k \). Then

\[
\prod_{i=1}^{k} f(x_i) \leq f(\lfloor \overline{x} \rfloor)^{k_1} f(\lceil \overline{x} \rceil)^{k_2}.
\]

**Proof.** It follows from induction on \( n \) that

\[
f(x)f(x + n) \leq f(x + 1)f(x + n - 1)
\]

for all \( x, n \in N \). To see this note that the inductive step is

\[
f(x)f(x + n + 1) = \frac{f(x)f(x + 2)f(x + 1)f(x + 1 + n)}{f(x + 1)f(x + 2)} \leq f(x + 1)f(x + n).
\]

By (2), if there exist \( p \) and \( q \) such that \( x_p - x_q > 1 \) and we let \( x'_p = x_p - 1, x'_q = x_q + 1, \) and \( x'_i = x_i \) for \( i \neq p, q \), then \( \prod_{i=1}^{k} f(x_i) \leq \prod_{i=1}^{k} f(x'_i) \). By repeating this step the upper bound eventually becomes \( f(\lfloor \overline{x} \rfloor)^{k_1} f(\lceil \overline{x} \rceil)^{k_2} \).
LEMMA 4. Let \( f(x) = x^{1/x} \) for \( x \in \mathbb{N} \). Suppose that \( n, p, r \in \mathbb{N}, \ n \geq 3, \) and \( p \leq (r - 1)/(n - 1) \). Then
\[
f(r - p - 1)^{r - pn + p - 1} f(r - p) (p + 1)^{n - p - r} \leq (r - 1)! r^{n/r - 1}.
\]

PROOF. Let \( k \in \mathbb{N}, \ k \leq p. \) Then \( r - k - 1 > 0. \) Therefore, by Lemma 2,
\[
f(r - k - 1) f(r - k + 1) \leq f(r - k)^2. \]
Since \( k \leq (r - 1)/(n - 1), \) we can raise both sides of this inequality to the power \( r - kn + k - 1 \) to obtain
\[
f(r - k - 1)^{r - kn + k - 1} f(r - k + 1)^{r - kn + k - 1} \leq f(r - k)^2 (r - kn + k - 1).
\]
This inequality is equivalent to
\[
f(r - k - 1)^{r - kn + k - 1} f(r - k) (k + 1)^{n - k - r}
\leq f(r - k)^{r - (k - 1)n + k - 2} f(r - k + 1)^{k n - k + 1 - r}.
\]
The desired inequality can now be obtained by successively applying (3) with \( k = p, \)
\( p - 1, \ p - 2, \ldots, 1. \)

LEMMA 5. Let \( x_1, x_2, \ldots, x_{n - 1}, \ r \in \mathbb{N} \) with \( n \geq 2. \) If \( \sum_{i=1}^{n-1} x_i \leq (n - 2)r + 1, \) then
\[
\prod_{i=1}^{n-1} x_i^{1/x_i} \leq (r - 1)! r^{n/r - 1}.
\]

PROOF. Since \( x^{1/x} \) is increasing in \( x \) we may assume that \( \sum_{i=1}^{n-1} x_i = (n - 2)r + 1. \) Hence
\[
\overline{x} = \frac{(n - 2)r + 1}{n - 1} = r - \frac{r - 1}{n - 1}.
\]
Let \( p = \lfloor (r - 1)/(n - 1) \rfloor. \) Then \( \lfloor \overline{x} \rfloor = \lfloor r - p \rfloor = r - p. \) If \( x \) is an integer and \( \lfloor \overline{x} \rfloor = r - p - 1, \)
\( \lfloor \overline{x} \rfloor = r - p. \) Otherwise. Since
\[
(r - pn + p - 1) (r - p - 1) + ((p + 1)n - p - r) (r - p) = (n - 2)r + 1
\]
we can apply Lemma 3 with \( f(x) = x^{1/x}, \ k = n - 1, \ k_1 = r - pn + p - 1, \) and
\( k_2 = (p + 1)n - p - r. \) (When \( \lfloor \overline{x} \rfloor = \lfloor \overline{x} \rfloor = r - p \) we have \( k_1 = 0. \) Therefore
Lemmas 3 and 4 imply that
\[
\prod_{i=1}^{n-1} x_i^{1/x_i} \leq f(r - p - 1)^{r - pn + p - 1} f(r - p) (p + 1)^{n - p - r} \leq (r - 1)! r^{n/r - 1}.
\]

LEMMA 6. If \( B = (b_{jk}) \) is a \((0,1)\)-matrix of order \( n \) with \( \sum_{j,k=1}^{n} b_{jk} = r, \) then
\[
\sum_{i=1}^{n-1} \sum_{j,k=1}^{n} b_{jk} \leq (n - 2)r + 1.
\]

PROOF.
\[
\sum_{i=1}^{n-1} \sum_{j,k=1}^{n} b_{jk} = \sum_{i=1}^{n-1} \left( r - \sum_{j=1}^{n} b_{ji} - \sum_{k=1}^{n} b_{ik} + b_{ii} \right)
\]
\[= (n - 3)r + \sum_{j=1}^{n} b_{jn} + \sum_{k=1}^{n} b_{nk} + \sum_{i=1}^{n-1} b_{ii}
\]
\[\leq (n - 3)r + r + 1 = (n - 2)r + 1.
\]
We now show how Schrijver's proof [4] of the Minc-Brègman bound can be altered to prove Theorem 1.

**Proof of Theorem 1.** The proof is by induction on \( n \). Clearly the theorem holds for \( n = 1 \). Let \( n > 1 \). It suffices to assume that \( r_i \in \mathbb{N} \) for \( i = 1, 2, \ldots, n \). Let \( A_{ijk} \) denote the 3-dimensional submatrix of \( A \) of order \( n - 1 \) obtained by deleting the three planes through cell \((i, j, k)\). Let

\[
S = \left\{ (\sigma, \tau) : \prod_{i=1}^{n} a_{i\sigma(i)} r(i) = 1 \right\}.
\]

Then \(|S| = \text{per} A\). Apply Lemma 1 with \( t = \text{per} A = \sum_{j,k;a_{ijk}=1} \text{per} A_{ijk} \) for \( i = 1, 2, \ldots, n \), to obtain

\[
(\text{per} A)^n \text{per} A = \prod_{i=1}^{n} (\text{per} A)^{\text{per} A} \leq \prod_{i=1}^{n} r_i^{\text{per} A} \prod_{j,k} (\text{per} A_{ijk})^{\text{per} A_{ijk}}.
\]

For each \( i \) we have

\[
\prod_{j,k \ a_{ijk}=1} (\text{per} A_{ijk})^{\text{per} A_{ijk}} = \prod_{j,k \ a_{ijk}=1} (\text{per} A_{ijk})^{\left\{ (\sigma, \tau) \in S : \sigma(i) = j, r(i) = k \right\}}
\]

\[
= \prod_{(\sigma, \tau) \in S} \text{per} A_{i\sigma(i)} r(i).
\]

By the induction hypothesis,

\[
\text{per} A_{i\sigma(i)} r(i) \leq \prod_{p: p \neq i} r(p, i)!^{1/r(p, i)}
\]

where

\[
r(p, i) = \sum_{j,k \ j \neq \sigma(i) \ k \neq r(i)} a_{pjk}.
\]

Combining the inequalities above we have

\[
(\text{per} A)^n \text{per} A \leq \prod_{i=1}^{n} r_i^{\text{per} A} \prod_{(\sigma, \tau) \in S} \prod_{p: p \neq i} r(p, i)!^{1/r(p, i)}
\]

\[
= \prod_{(\sigma, \tau) \in S} \left( \prod_{i=1}^{n} r_i \right)^{n} \prod_{p=1}^{n} \prod_{i \neq p} r(p, i)!^{1/r(p, i)}.
\]
Fix $\sigma, r,$ and $p$ and let $b_{jk} = a_{p\sigma(j)\tau(k)}$. Then $\sum_{j,k} b_{jk} = r_p$ and for each $i$,

$$\sum_{j,k}^{j,k \neq i, k \neq i} r(p, i) = r(p, i).$$

Hence, by Lemma 6,

$$\sum_{i \neq p} r(p, i) \leq (n - 2) r_p + 1.$$

Therefore, by Lemma 5, if $n > 2$, then

$$\prod_{i \neq p} r(p, i)^{1/r(p, i)} \leq (r_p - 1)! r_p^{1/n} r_p^{-1}.$$

Moreover, it is easy to see that this inequality also holds for $n = 2$. Therefore,

$$(\text{per } A)^{n \text{ per } A} \leq \prod_{(\sigma, r) \in S} \left( \prod_{i=1}^{n} r_i \right) \prod_{p=1}^{n} (r_p - 1)! r_p^{1/n} r_p^{-1}$$

and the theorem follows.

For some values of the $r_i$ it is easy to construct matrices $A$ such that equality holds in Theorem 1. Let $A_m = (a_{ijk})$ denote the 3-dimensional $(0,1)$-matrix of order $m$ with $a_{ij} = 1$ for $i, j = 1, 2, \ldots, m$, and $a_{ijk} = 0$ otherwise. Then $r_i = \sum_{j,k=1}^{n} a_{ijk} = m$ for $i = 1, 2, \ldots, m$, and per $A_m = m!$. Now let $n_1, n_2, \ldots, n_t$ be a partition of $n$, and let $A$ be the direct sum of $A_{n_1}, A_{n_2}, \ldots, A_{n_t}$. It follows that $A$ is a 3-dimensional $(0, 1)$-matrix of order $n$ such that equality holds in Theorem 1.

3. The 2-permanent. Theorem 1 can be used to obtain another extension of the Minc-Brègman inequality to 3-dimensional matrices. Let $A = (a_{ijk})$ be a 3-dimensional $(0, 1)$-matrix of order $n$. The planes of $A$ are the submatrices obtained by fixing one of $i, j, k$, and the lines of $A$ are the submatrices obtained by fixing two of $i, j, k$. Observe that the permanent (or 1-permanent) of $A$ is equal to the summation of all products of $n$ entries of $A$, no two entries from the same plane. Similarly, the 2-permanent of $A$ is defined to be the summation of all products of $n^2$ entries of $A$, no two entries from the same line [2]. We have the following upper bound for the 2-permanent of a $(0, 1)$-matrix.

THEOREM 2. Let $A = (a_{ijk})$ be a 3-dimensional $(0, 1)$-matrix of order $n$, and let $r_{ij} = \sum_{k=1}^{n} a_{ijk}$ for $i, j = 1, 2, \ldots, n$. Then

$$2\text{-per } A \leq \prod_{i,j=1}^{n} r_{ij}^{1/r_{ij}}.$$

PROOF. As shown by the authors [2], there exists a 3-dimensional $(0, 1)$-matrix $B = (b_{ijk})$ of order $n^2$ such that per $B = 2$-per $A$. Moreover, $B$ can be chosen so that if $s_i = \sum_{j,k=1}^{n^2} b_{ijk}$ for $i = 1, 2, \ldots, n^2$, then $\{s_i : i = 1, 2, \ldots, n^2\} = \{r_{ij} : i, j = 1, 2, \ldots, n\}$. Therefore, if Theorem 1 is applied to $B$ the desired upper bound on 2-per $A$ is obtained.
REFERENCES


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