AN UPPER BOUND FOR THE PERMANENT OF A 3-DIMENSIONAL (0,1)-MATRIX

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ABSTRACT. Let $A = (a_{ijk})$ be a 3-dimensional matrix of order $n$. The permanent of $A$ is defined by

$$\text{per } A = \sum_{\sigma, \tau \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)\tau(i)},$$

where $S_n$ is the symmetric group on $\{1, 2, \ldots, n\}$. Suppose that $A$ is a (0,1)-matrix and that $r_i = \sum_{j,k=1}^{n} a_{ijk}$ for $i = 1, 2, \ldots, n$. In this paper it is shown that $\text{per } A \leq \prod_{i=1}^{n} r_i^{1/r_i}$. A similar bound is then obtained for a second function, the 2-permanent of a 3-dimensional matrix, that is another analogue of the permanent of an ordinary (2-dimensional) matrix.

1. Introduction. It was conjectured by Minc [3] and proved by Brègman [1] that if $A$ is a (0,1)-matrix of order $n$ with row sums $r_1, r_2, \ldots, r_n$, then

$$\text{per } A \leq \prod_{i=1}^{n} r_i^{1/r_i},$$

where by definition $0^{1/0} = 0$. In this paper it is shown that an analogous result holds for 3-dimensional matrices. The permanent of a 3-dimensional matrix $A = (a_{ijk})$ of order $n$ is defined by

$$\text{per } A = \sum_{\sigma, \tau \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)\tau(i)}$$

where $S_n$ is the symmetric group on $\{1, 2, \ldots, n\}$. We now state our principal result.

THEOREM 1. Let $A = (a_{ijk})$ be a 3-dimensional (0,1)-matrix of order $n$, and let $r_i = \sum_{j,k=1}^{n} a_{ijk}$ for $i = 1, 2, \ldots, n$. Then

$$\text{per } A \leq \prod_{i=1}^{n} r_i^{1/r_i}.$$

The proof of Theorem 1, presented in §2, is an adaptation of Schrijver’s proof [4] of the Minc-Brègman bound, with the extension to 3-dimensional matrices requiring a number of additional lemmas which may be of some independent interest.

In equation (1) and Theorem 1 the planes of $A$ play the role of the rows and columns of an ordinary (2-dimensional) matrix, where a plane of a 3-dimensional
matrix $A = (a_{ijk})$ is a submatrix obtained by fixing one of the indices $i, j, k$ and letting the other two vary. We could also have taken the analogue of rows and columns to be the lines of $A$, obtained by fixing two of these indices. In §3 we give a second definition of the permanent of a 3-dimensional matrix, called the 2-permanent, based on this observation. Using a result from [2] which relates these two types of permanents we obtain an upper bound for the 2-permanent of a 3-dimensional $(0, 1)$-matrix similar to the one given in Theorem 1, with plane sums replaced by line sums.

2. Proof of the principal result. We write $N$, $Z$, and $R^+$ to denote, respectively, the set of natural numbers, the set of integers, and the set of positive real numbers. Also, for any real number $x$, we use the notation $\lfloor x \rfloor = \max\{k \in Z : k \leq x\}$ and $\lceil x \rceil = \min\{k \in Z : x < k\}$. The first lemma appears in [4].

**Lemma 1.** If $t = \sum_{i=1}^{n} t_i$, $t_i \in R^+$, then $t^t \leq n^t \prod_{i=1}^{n} t_i^t$.

**Lemma 2.** For all $n \in N$, $n^{1/n} (n + 2)^{1/(n+2)} \leq (n + 1)^{2/(n+1)}$.

**Proof.** It is well known that if $x_1, x_2, \ldots, x_k \in R^+$, then $\prod_{i=1}^{k} x_i \leq \bar{x}^k$ where $\bar{x} = \left( \sum_{i=1}^{k} x_i \right)/k$. Let $x_i = i$ for $i = 1, 2, \ldots, n$, $x_i = i-n$ for $i = n+1, n+2, \ldots, 2n$, and $x_i = n+2$ for $i = 2n+1, 2n+2, \ldots, n^2+3n$. Then

$$
\prod_{i=1}^{n^2+3n} x_i = n!^2 (n + 2)^{n^2+n}
$$

and

$$
\sum_{i=1}^{n^2+3n} x_i = (n^2 + 3n)(n + 1).
$$

Hence the inequality above becomes $n!^2 (n + 2)^{n^2+n} \leq (n + 1)^{n^2+3n}$, which is equivalent to the desired inequality.

**Lemma 3.** Suppose that a function $f: N \rightarrow R^+$ has the property that

$$
f(x)f(x + 2) \leq (f(x + 1))^2
$$

for all $x \in N$, and that $x_1, x_2, \ldots, x_k \in N$. Let $\sum_{i=1}^{k} x_i = k\bar{x} = k\lfloor \bar{x} \rfloor + k_2 \lceil \bar{x} \rceil$, where $k_1, k_2 \in Z$, $k_1, k_2 \geq 0$, and $k_1 + k_2 = k$. Then

$$
\prod_{i=1}^{k} f(x_i) \leq f(\lfloor \bar{x} \rfloor)^{k_1} f(\lceil \bar{x} \rceil)^{k_2}.
$$

**Proof.** It follows from induction on $n$ that

(2) $f(x)f(x + n) \leq f(x + 1)f(x + n - 1)$

for all $x, n \in N$. To see this note that the inductive step is

$$
f(x)f(x + n + 1) = \frac{f(x)f(x + 2)f(x + 1)f(x + n)}{f(x + 1)f(x + 2)} \leq f(x + 1)f(x + n).
$$

By (2), if there exist $p$ and $q$ such that $x_p - x_q > 1$ and we let $x'_p = x_p - 1$, $x'_q = x_q + 1$, and $x'_i = x_i$ for $i \neq p, q$, then $\prod_{i=1}^{k} f(x_i) \leq \prod_{i=1}^{k} f(x'_i)$. By repeating this step the upper bound eventually becomes $f(\lfloor \bar{x} \rfloor)^{k_1} f(\lceil \bar{x} \rceil)^{k_2}$.
LEMMA 4. Let $f(x) = x^{1/x}$ for $x \in \mathbb{N}$. Suppose that $n, p, r \in \mathbb{N}$, $n \geq 3$, and $p \leq (r - 1)/(n - 1)$. Then

$$f(r - p - 1)^{r - pn + p - 1}f(r - p)^{(p + 1)n - p - r} \leq (r - 1)!r^{n/r - 1}.$$

PROOF. Let $k \in \mathbb{N}$, $k \leq p$. Then $r - k - 1 > 0$. Therefore, by Lemma 2, $f(r - k - 1)f(r - k + 1) \leq f(r - k)^2$. Since $k \leq (r - 1)/(n - 1)$, we can raise both sides of this inequality to the power $r - kn + k - 1$ to obtain

$$f(r - k - 1)^{r - kn + k - 1}f(r - k + 1)^{r - kn + k - 1} \leq f(r - k)^2(r - kn + k - 1).$$

This inequality is equivalent to

$$f(r - k - 1)^{r - kn + k - 1}f(r - k)^{(k + 1)n - k - r} \leq f(r - k)^{r - (k - 1)n + k - 2}(r - k + 1)^{kn - k + 1 - r}.$$

The desired inequality can now be obtained by successively applying (3) with $k = p, p - 1, p - 2, \ldots, 1$.

LEMMA 5. Let $x_1, x_2, \ldots, x_{n-1}, r \in \mathbb{N}$ with $n > 2$. If $\sum_{i=1}^{n-1} x_i \leq (n - 2)r + 1$, then

$$\prod_{i=1}^{n-1} x_i^{1/x_i} \leq (r - 1)!r^{n/r - 1}.$$

PROOF. Since $x^{1/x}$ is increasing in $x$ we may assume that $\sum_{i=1}^{n-1} x_i = (n - 2)r + 1$. Hence

$$\bar{x} = \frac{(n - 2)r + 1}{n - 1} = r - \frac{r - 1}{n - 1}.$$ Let $p = [(r - 1)/(n - 1)]$. Then $[\bar{x}] = \lfloor \bar{x} \rfloor = r - p$ if $x$ is an integer and $[\bar{x}] = r - p - 1$, $\lfloor \bar{x} \rfloor = r - p$ otherwise. Since

$$(r - pn + p - 1)(r - p - 1) + ((p + 1)n - p - r)(r - p) = (n - 2)r + 1$$

we can apply Lemma 3 with $f(x) = x^{1/x}$, $k = n - 1$, $k_1 = r - pn + p - 1$, and $k_2 = (p + 1)n - p - r$. (When $\lfloor \bar{x} \rfloor = \lceil \bar{x} \rceil = r - p$ we have $k_1 = 0$.) Therefore Lemmas 3 and 4 imply that

$$\prod_{i=1}^{n-1} x_i^{1/x_i} \leq f(r - p - 1)^{r - pn + p - 1}f(r - p)^{(p + 1)n - p - r} \leq (r - 1)!r^{n/r - 1}.$$

LEMMA 6. If $B = (b_{jk})$ is a $(0, 1)$-matrix of order $n$ with $\sum_{j, k=1}^{n} b_{jk} = r$, then

$$\sum_{i=1}^{n-1} \sum_{j, k \neq i} b_{jk} \leq (n - 2)r + 1.$$ 

PROOF.

$$\sum_{i=1}^{n-1} \sum_{j, k \neq i} b_{jk} = \sum_{i=1}^{n-1} \left( r - \sum_{j=1}^{n} b_{ji} - \sum_{k=1}^{n} b_{ik} + b_{ii} \right)$$

$$= (n - 3)r + \sum_{j=1}^{n} b_{jn} + \sum_{k=1}^{n} b_{nk} + \sum_{i=1}^{n-1} b_{ii}$$

$$\leq (n - 3)r + r + 1 = (n - 2)r + 1.$$
We now show how Schrijver’s proof [4] of the Minc-Brègman bound can be altered to prove Theorem 1.

**Proof of Theorem 1.** The proof is by induction on \( n \). Clearly the theorem holds for \( n = 1 \). Let \( n > 1 \). It suffices to assume that \( r_i \in \mathbb{N} \) for \( i = 1, 2, \ldots, n \). Let \( A_{ijk} \) denote the 3-dimensional submatrix of \( A \) of order \( n - 1 \) obtained by deleting the three planes through cell \((i,j,k)\). Let

\[
S = \left\{ (\sigma, \tau) : \prod_{i=1}^{n} a_{i\sigma(i)r(i)} = 1 \right\}.
\]

Then \( |S| = \text{per} A \). Apply Lemma 1 with \( t = \text{per} A = \sum_{j,k;a_{ijk}=1} \text{per} A_{ijk} \) for \( i = 1, 2, \ldots, n \), to obtain

\[
\left( \text{per} A \right)^n \text{per} A = \prod_{i=1}^{n} (\text{per} A)_{\text{per} A}^n \\
\leq \prod_{i=1}^{n} r_i \text{per} A \prod_{j,k;a_{ijk}=1} (\text{per} A_{ijk})_{\text{per} A_{ijk}}.
\]

For each \( i \) we have

\[
\prod_{j,k;a_{ijk}=1} (\text{per} A_{ijk})_{\text{per} A_{ijk}} = \prod_{j,k;a_{ijk}=1} (\text{per} A_{ijk})_{\{ (\sigma, \tau) \in S : \sigma(i)=j, \tau(i)=k \}}
\]

\[
= \prod_{(\sigma, \tau) \in S} \text{per} A_{i\sigma(i)r(i)}.
\]

By the induction hypothesis,

\[
\text{per} A_{i\sigma(i)r(i)} \leq \prod_{p \neq i} r(p, i)!^{1/r(p, i)}
\]

where

\[
r(p, i) = \sum_{j,k : j \neq \sigma(i), k \neq \tau(i)} a_{pjk}.
\]

Combining the inequalities above we have

\[
\left( \text{per} A \right)^n \text{per} A \leq \prod_{i=1}^{n} r_i \text{per} A \prod_{(\sigma, \tau) \in S} \prod_{p \neq i} r(p, i)!^{1/r(p, i)}
\]

\[
= \prod_{(\sigma, \tau) \in S} \left( \prod_{i=1}^{n} r_i \right) \prod_{i=1}^{n} \prod_{p \neq i} r(p, i)!^{1/r(p, i)}
\]

\[
= \prod_{(\sigma, \tau) \in S} \left( \prod_{i=1}^{n} r_i \right) \prod_{p=1}^{n} \prod_{i \neq p} r(p, i)!^{1/r(p, i)}.
\]
Fix \( \sigma, \tau, \) and \( p \) and let \( b_{jk} = a_{p\sigma(j)\tau(k)} \). Then \( \sum_{j,k} b_{jk} = r_p \) and for each \( i \),

\[
\sum_{j,k \neq i, k \neq i} r(p, i) = r_p.
\]

Hence, by Lemma 6,

\[
\sum_{i \neq p} r(p, i) \leq (n - 2)r_p + 1.
\]

Therefore, by Lemma 5, if \( n > 2 \), then

\[
\prod_{i \neq p} r(p, i)^{1/r(p, i)} \leq (r_p - 1)!r_p^{1/n/r_p - 1}.
\]

Moreover, it is easy to see that this inequality also holds for \( n = 2 \). Therefore,

\[
\frac{(\text{per } A)^n}{\text{per } A} \leq \prod_{(\sigma, \tau) \in S} \left( \prod_{i=1}^{n} r_i \right) \prod_{p=1}^{n} (r_p - 1)!r_p^{1/n/r_p - 1}
\]

\[
= \prod_{(\sigma, \tau) \in S} \prod_{p=1}^{n} r_p^{1/n/r_p} = \left( \prod_{i=1}^{n} r_i^{1/r_i} \right)^n
\]

and the theorem follows.

For some values of the \( r_i \) it is easy to construct matrices \( A \) such that equality holds in Theorem 1. Let \( A_m = (a_{ijk}) \) denote the 3-dimensional \((0, 1)\)-matrix of order \( m \) with \( a_{ij} = 1 \) for \( i, j = 1, 2, \ldots, m \), and \( a_{ijk} = 0 \) otherwise. Then \( r_i = \sum_{j,k=1}^{n} a_{ijk} = m \) for \( i = 1, 2, \ldots, m \), and \( \text{per } A_m = m! \). Now let \( n_1, n_2, \ldots, n_t \) be a partition of \( n \), and let \( A \) be the direct sum of \( A_{n_1}, A_{n_2}, \ldots, A_{n_t} \). It follows that \( A \) is a 3-dimensional \((0, 1)\)-matrix of order \( n \) such that equality holds in Theorem 1.

3. The 2-permanent. Theorem 1 can be used to obtain another extension of the Minc-Brègman inequality to 3-dimensional matrices. Let \( A = (a_{ijk}) \) be a 3-dimensional \((0, 1)\)-matrix of order \( n \). The planes of \( A \) are the submatrices obtained by fixing one of \( i, j, k \), and the lines of \( A \) are the submatrices obtained by fixing two of \( i, j, k \). Observe that the permanent (or 1-permanent) of \( A \) is equal to the summation of all products of \( n \) entries of \( A \), no two entries from the same plane. Similarly, the 2-permanent of \( A \) is defined to be the summation of all products of \( n^2 \) entries of \( A \), no two entries from the same line \([2]\). We have the following upper bound for the 2-permanent of a \((0, 1)\)-matrix.

**Theorem 2.** Let \( A = (a_{ijk}) \) be a 3-dimensional \((0, 1)\)-matrix of order \( n \), and let \( r_{ij} = \sum_{k=1}^{n} a_{ijk} \) for \( i, j = 1, 2, \ldots, n \). Then

\[
2\text{-per } A \leq \prod_{i,j=1}^{n} r_{ij}^{1/r_{ij}}.
\]

**Proof.** As shown by the authors \([2]\), there exists a 3-dimensional \((0, 1)\)-matrix \( B = (b_{ijk}) \) of order \( n^2 \) such that \( \text{per } B = 2\text{-per } A \). Moreover, \( B \) can be chosen so that if \( s_i = \sum_{j,k=1}^{n^2} b_{ijk} \) for \( i = 1, 2, \ldots, n^2 \), then \( \{s_i : i = 1, 2, \ldots, n^2\} = \{r_{ij} : i, j = 1, 2, \ldots, n\} \). Therefore, if Theorem 1 is applied to \( B \) the desired upper bound on 2-per \( A \) is obtained.
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