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BEST L_1-APPROXIMATION WITH VARYING WEIGHTS

ANDRÁS KROÓ

ABSTRACT. It is proved in this note that the so-called A-property is necessary in order that the finite-dimensional space U be Chebyshev in $C(K)$ with respect to the norm $\|f\| = \int_K \omega|f|$ for *every* positive continuous weight ω. It is also shown that for each finite-dimensional subspace U there exists a positive continuous weight ω such that U is Chebyshev in $C(K)$ with respect to this weight ω.

NOTATION. Let K be a compact subset of $\mathbf{R}^s$ and denote by $C(K)$ the set of continuous real functions on K. Furthermore, we denote $W_C = \{\omega \in C(K) \colon \omega > 0 \text{ on } K\}$ and $W = \{\omega \colon \omega \text{ is measurable and } 0 < \inf\{\omega(x) \colon x \in K\} \leq \sup\{\omega(x) \colon x \in K\} < \infty\}$. For any $\omega \in W$ we denote by $C_\omega(K)$ the space $C(K)$ endowed with the norm

$$\|f\|_{L_\omega(K)} = \int_K \omega|f|. \tag{1}$$

Furthermore, we shall say that a finite-dimensional subspace U in $C(K)$ is a Chebyshev subspace of $C_\omega(K)$ if every $f \in C(K)$ has a unique best approximant in U with respect to norm (1). We shall assume that $\overline{\operatorname{Int} K} = K$.

By a result of Strauss [**3**] if U satisfies the so-called A-property then U is a Chebyshev subspace of $C_\omega(K)$ for every $\omega \in W$. In [**1**] we proved the converse of this statement: If U is Chebyshev in $C_\omega(K)$ for every $\omega \in W$ then U is an A-space. Independently, Pinkus [**2**] gave another version of this result. He showed using a different method that the A-property is necessary even for uniqueness with respect to all continuous weights $\omega \in W_C$. However, a price had to be paid in that U should satisfy

$$\mu(Z(u)) = \mu(\operatorname{Int}(Z(u))), \qquad u \in U. \tag{2}$$

(Here and in what follows $Z(u) = \{x \in K \colon u(x) = 0\}$, $\mu(S)$ denotes the Lebesgue measure of S.) This left open the question whether the A-property is necessary for uniqueness for every $\omega \in W_C$ without any assumption on U. In this note we give an affirmative answer to this question showing that restriction (2) can be removed.

THEOREM 1. *If U is a Chebyshev subspace of $C_\omega(K)$ for every $\omega \in W_C$ then U is an A-space.*

In the proof of Theorem 1 we shall follow the approach used in [**1**] with some more technical details needed for construction of continuous weight.

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The above results indicate that A-spaces are the only subspaces of $C(K)$ which guarantee uniqueness of best approximation in $C_\omega(K)$ for *every* $\omega \in W_C$. On the other hand it will be shown below that *each* finite-dimensional subspace of $C(K)$ is Chebyshev in $C_\omega(K)$ for *some* $\omega \in W_C$.

THEOREM 2. *For any finite-dimensional subspace U in $C(K)$ there exists an $\omega \in W_C$ such that U is a Chebyshev subspace of $C_\omega(K)$.*

First of all let us recall the definition of A-spaces. Denote $U^* = \{u^* \in C(K)$: there exists $u \in U$ such that $|u| = |u^*|$ on $K\}$.

DEFINITION. The finite-dimensional subspace U in $C(K)$ is called an A-space if for every $u^* \in U^*$ there exists a $u \in U\backslash\{0\}$ such that $uu^* \geq 0$ on K and $u = 0$ a.e. on $Z(u^*)$.

We shall use in our considerations the following result of Strauss [**4**].

LEMMA 1. *U is a Chebyshev subspace of $C_\omega(K)$ ($\omega \in W$) if and only if for every $u^* \in U^*\backslash\{0\}$ there exists a $u \in U$ such that*

$$\int_{K\backslash Z(u^*)} \omega u \operatorname{sgn} u^* > \int_{Z(u^*)} \omega |u|.$$

In order to prove Theorem 1 we need some further lemmas. As usual $\|g\|_{C(D)}$ denotes supremum norm on $D \subset \mathbf{R}^s$.

Let $D \subset \mathbf{R}^s$ be open and bounded, $M > 0$. Set $W_M(D) = \{\omega \in C(\overline{D})\colon \omega > 0$ on $\overline{D}$, $\omega = M$ on $\operatorname{Bd} D$ and $\int_D \omega \leq 1\}$.

LEMMA 2. *Let $g \in C(D)$, $\|g\|_{C(D)} < \infty$, $g \geq 0$ on D and $\int_D g < 1$. For any $\varepsilon > 0$ there exists $\omega \in W_M(D)$ such that $\|g - \omega\|_{L_1(D)} < \varepsilon$.*

PROOF. Without loss of generality we may assume that $g > 0$ on D. Denote $\rho(x) = \operatorname{dist}(x, \operatorname{Bd} D)$ and set

$$\Psi_\varepsilon(x) = \begin{cases} 1 - \rho(x)/\varepsilon, & \text{if } \rho(x) < \varepsilon, \\ 0, & \text{if } \rho(x) \geq \varepsilon, \end{cases} \qquad \varphi_\varepsilon(x) = \begin{cases} \rho(x)/\varepsilon, & \text{if } \rho(x) < \varepsilon, \\ 1, & \text{if } \rho(x) \geq \varepsilon. \end{cases}$$

Evidently, Ψ_ε, $\varphi_\varepsilon \in C(\overline{D})$, Ψ_ε, $\varphi_\varepsilon \geq 0$ on $\overline{D}$. Consider the function $g_\varepsilon(x) = M\Psi_\varepsilon(x) + g(x)\varphi_\varepsilon(x)$. Then $g_\varepsilon \in C(\overline{D})$, $g_\varepsilon > 0$ on $\overline{D}$ and $g_\varepsilon = M$ on $\operatorname{Bd} D$. Furthermore, since $g_\varepsilon = g$ if $\rho(x) \geq \varepsilon$ we have

$$\begin{aligned} \|g - g_\varepsilon\|_{L_1(D)} &= \int_{\{x \in D\colon \rho(x) < \varepsilon\}} |g - g_\varepsilon| \\ &\leq (M + 2\|g\|_{C(D)}) \cdot \mu\{x \in D\colon \rho(x) < \varepsilon\} \to 0 \qquad (\varepsilon \to 0). \end{aligned}$$

Thus, in particular, $\int_D g_\varepsilon \to \int_D g < 1$ ($\varepsilon \to 0$). This implies that, for ε sufficiently small, $\int_D g_\varepsilon \leq 1$ and, consequently, $g_\varepsilon \in W_M(D)$. This completes the proof of the lemma.

LEMMA 3. *Let $g \in C(D)$, $\|g\|_{C(D)} < \infty$ and assume that $\int_D \omega g \geq 0$ for every $\omega \in W_M(D)$. Then $g \geq 0$ on D.*

PROOF. Let $\tilde{g}(x)$ be equal to 0 if $g(x) \geq 0$ and $\tilde{g}(x) = -g(x)$ if $g(x) < 0$. Obviously, $\tilde{g} \in C(D)$, $\|\tilde{g}\|_{C(D)} < \infty$ and $\tilde{g} \geq 0$ on D. We may assume that $\int_D |g| < 1$ which implies that $\int_D \tilde{g} < 1$. Thus we can apply Lemma 1 to $\tilde{g}$; that is

we can find an $\tilde{\omega} \in W_M(D)$ such that $\|\tilde{g} - \tilde{\omega}\|_{L_1(D)} < \varepsilon$. Using that $\int_D \tilde{\omega} g \geq 0$ and setting $D_1 = \{x \in D : g(x) < 0\}$, we obtain

$$\begin{aligned}\int_{D_1} g^2 &\leq \int_{D_1} |g\tilde{\omega} + g^2| - \int_{D_1} g\tilde{\omega} = \int_{D_1} |g\tilde{\omega} + g^2| - \int_D g\tilde{\omega} + \int_{D\setminus D_1} g\tilde{\omega} \\ &\leq \int_{D_1} |g\tilde{\omega} + g^2| + \int_{D\setminus D_1} |g|\tilde{\omega} \leq \|g\|_{C(D)} \left\{ \int_{D_1} |\tilde{\omega} + g| + \int_{D\setminus D_1} \tilde{\omega} \right\} \\ &= \|g\|_{C(D)} \|\tilde{\omega} - \tilde{g}\|_{L_1(D)} < \varepsilon \|g\|_{C(D)}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary this yields that $\int_{D_1} g^2 = 0$, i.e. $\mu(D_1) = 0$. This and continuity of g on D imply that $g \geq 0$ on D.

PROOF OF THEOREM 1. Consider an arbitrary $u^* \in U^* \setminus \{0\}$. By Lemma 1 for every $\omega \in W_C$ there exists a $u \in U$ such that

$$\int_{K\setminus Z(u^*)} \omega u \operatorname{sgn} u^* > \int_{Z(u^*)} \omega |u|. \tag{3}$$

Set $\tilde{U} = \{u \in U : u = 0 \text{ a.e. on } Z(u^*)\}$, and let $U = \tilde{U} \oplus U_1$. This means that U_1 is a linear subspace of U and elements of U_1 cannot vanish a.e. on $Z(u^*)$. Hence there exists a constant $M > 0$ such that

$$\|u_1\|_{C(K)} \leq M \int_{Z(u^*)} |u_1| \tag{4}$$

holds for every $u_1 \in U_1$. Set $D = K \setminus Z(u^*)$ and consider the set $W_M(D)$. Let $\varphi_1, \ldots, \varphi_r$ be a basis in $\tilde{U}$. We introduce the set

$$A_r = \left\{ \left(\int_D \omega \varphi_i \operatorname{sgn} u^* \right)_{i=1}^r : \omega \in W_M(D) \right\}.$$

Evidently, A_r is a convex set in R^r. Furthermore, Lemma 2 implies that $0 \in \overline{A}_r$. We claim that $0 \notin A_r$. Assume that in contradiction $0 \in A_r$. Then for some $\tilde{\omega} \in W_M(D)$

$$\int_D \tilde{\omega} u \operatorname{sgn} u^* = 0, \qquad u \in \tilde{U}. \tag{5}$$

Extend $\tilde{\omega}$ to $Z(u^*)$ setting $\tilde{\omega} = M$ on $Z(u^*)$. Obviously, $\tilde{\omega} \in W_C$. Consider an arbitrary $u = u_1 + \tilde{u} \in U$, where $u_1 \in U_1$, $\tilde{u} \in \tilde{U}$. Then by (5) and (4)

$$\begin{aligned}\left| \int_D \tilde{\omega} u \operatorname{sgn} u^* \right| &= \left| \int_D \tilde{\omega} u_1 \operatorname{sgn} u^* \right| \\ &\leq \|u_1\|_{C(K)} \int_D \tilde{\omega} \leq \|u_1\|_{C(K)} \leq M \int_{Z(u^*)} |u_1| = \int_{Z(u^*)} \tilde{\omega} |u|.\end{aligned}$$

But this contradicts (3). Hence $0 \notin A_r$, i.e. $0 \in \operatorname{Bd} A_r$. Then there exists a hyperplane supporting A_r at 0; that is, for some $u \in \tilde{U} \setminus \{0\}$ we have that

$$\int_D \omega u \operatorname{sgn} u^* \geq 0$$

for every $\omega \in W_M(D)$. This and Lemma 3 imply that $uu^* \geq 0$ on D, and therefore $uu^* \geq 0$ on K, as well. Moreover, $u = 0$ a.e. at $Z(u^*)$, and thus the A-property of U is verified.

PROOF OF THEOREM 2. Let $U = \operatorname{span}\{u_1, \dots, u_n\}$. Since $K = \overline{\operatorname{Int} K}$ it follows that $u_1, \dots, u_n$ are linearly independent on $\operatorname{Int} K$. Therefore, there exist points $x_1, \dots, x_n \in \operatorname{Int} K$ such that

$$\det \begin{pmatrix} u_1(x_1), \dots, u_1(x_n) \\ \dots \\ u_n(x_1), \dots, u_n(x_n) \end{pmatrix} \neq 0. \tag{6}$$

Denote by $B_r(x)$ the closed ball with center at x and radius r. Set

$$D_1^{(m)} = \bigcup_{i=1}^{n} B_{1/m}(x_i); \qquad D_2^{(m)} = \bigcup_{i=1}^{n} B_{1/m+1/m^2}(x_i).$$

For a suitable m_0 we have that $D_1^{(m)} \subset D_2^{(m)} \subset \operatorname{Int} K$ if $m \geq m_0$. Consider a weight ω_m equal to 1 and $1/m^{s+1}$ on $D_1^{(m)}$ and $K \backslash D_2^{(m)}$, respectively. We can easily extend ω_m to $D_2^{(m)} \backslash D_1^{(m)}$ in such a way that $\omega_m \in W_C$ and $\omega_m \leq 1$ on K. Assume that the statement of the theorem is false. Then, in particular, U is not Chebyshev in $C_{\omega_m}(K)$ for every $m \geq m_0$. It follows from Lemma 1 that there exists $u_m^* \in U^* \backslash \{0\}$, $\|u_m^*\|_{C(K)} = 1$ such that, for every $u \in U$,

$$\left| \int_{K \backslash Z(u_m^*)} \omega_m u \operatorname{sgn} u_m^* \right| \leq \int_{Z(u_m^*)} \omega_m |u| \qquad (m \geq m_0). \tag{7}$$

By definition of U^* there exists $u_m \in U$ satisfying $|u_m^*| = |u_m|$ on K. In particular, we obtain that $\|u_m\|_{C(K)} = 1$. Therefore, without loss of generality we may assume that $u_m \to u_0 \in U$ in $C(K)$-norm, $\|u_0\|_{C(K)} = 1$. By (6), $u_0(x_r) \neq 0$ for at least one $1 \leq r \leq n$. Moreover, (6) also implies that there exists $\tilde{u} \in U \backslash \{0\}$ such that $\tilde{u}(x_r) = 1$ and $\tilde{u}(x_i) = 0$ for $1 \leq i \leq n$, $i \neq r$. Applying (7) to $\tilde{u}$ we have

$$\left| \int_{K \backslash Z(u_m^*)} \omega_m \tilde{u} \operatorname{sgn} u_m^* \right| \leq \int_{Z(u_m^*)} \omega_m |\tilde{u}|, \qquad m \geq m_0. \tag{8}$$

Since $u_0(x_r) \neq 0$ and $|u_m^*| = |u_m| \to |u_0|$ $(m \to \infty)$ uniformly on K it follows that $B_{1/m}(x_r) \subset K \backslash Z(u_m^*)$ for $m \geq m_1$ with sufficiently large m_1. Therefore, we obtain by (8)

$$\begin{aligned} \left| \int_{B_{1/m}(x_r)} \tilde{u} \right| &= \left| \int_{B_{1/m}(x_r)} \omega_m \tilde{u} \operatorname{sgn} u_m^* \right| \leq \int_{K \backslash B_{1/m}(x_r)} \omega_m |\tilde{u}| \\ &= \sum_{\substack{i=1 \\ i \neq r}}^{n} \int_{B_{1/m}(x_i)} \omega_m |\tilde{u}| + \int_{D_2^{(m)} \backslash D_1^{(m)}} \omega_m |\tilde{u}| + \int_{K \backslash D_2^{(m)}} \omega_m |\tilde{u}|. \end{aligned} \tag{9}$$

Furthermore, we have

$$\int_{K \backslash D_2^{(m)}} \omega_m |\tilde{u}| = \frac{1}{m^{s+1}} \int_{K \backslash D_2^{(m)}} |\tilde{u}| \leq \frac{\mu(K) \|\tilde{u}\|_{C(K)}}{m^{s+1}}, \tag{10}$$

$$\int_{D_2^{(m)} \backslash D_1^{(m)}} \omega_m |\tilde{u}| \leq \int_{D_2^{(m)} \backslash D_1^{(m)}} |\tilde{u}| \leq \|\tilde{u}\|_{C(K)} \mu(D_2^{(m)} \backslash D_1^{(m)}) \leq \frac{a_s \|\tilde{u}\|_{C(K)}}{m^{s+1}} \tag{11}$$

with a constant $a_s > 0$ depending only on s. Finally, using that $\tilde{u}(x_i) = 0$ for $1 \le i \le n$, $i \ne r$, we obtain

$$\sum_{\substack{i=1\\ i\ne r}}^{n} \int_{B_{1/m}(x_i)} \omega_m |\tilde{u}| = \sum_{\substack{i=1\\ i\ne r}}^{n} \int_{B_{1/m}(x_i)} |\tilde{u}| \le \frac{C_s(n-1)\omega(\tilde{u}, 1/m)}{m^s}, \tag{12}$$

where $C_s > 0$ depends only on s and $\omega(\tilde{u}, h)$ is the uniform modulus of continuity of $\tilde{u}$. Applying inequalities (10)–(12) in (9) leads to

$$\left| \int_{B_{1/m}(x_r)} \tilde{u} \right| \le o(m^{-s}), \qquad m \ge m_1,$$

which in turn implies that $\tilde{u}(x_r) = 0$, a contradiction. The theorem is proved.

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Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Reàltanoda u. 13-15, H-1053 Hungary

Current address: Department of Mathematics, University of South Florida, Tampa, Florida 33620