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GUIDING FUNCTIONS AND PERIODIC SOLUTIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. A new definition of a guiding function for functional differential equations is given, which is sometimes better for applications than the known one by Mawhin. We then prove an existence result for periodic solutions of FDEs related to the new definition.

1. Introduction. In this paper we use techniques related to guiding functions in order to prove the existence of periodic solutions to the functional differential equation

$$x'(t) = q(t, x_t). \tag{1}$$

The idea of a guiding function goes back to Krasnosel'skii [**3**], who introduced it first in order to study the periodic solutions of an ordinary differential equation. Krasnosel'skii [**4**] later extended his definition of a guiding function in order to include some delay differential equations. Subsequently Mawhin, first in [**7**] and then in [**2, 8, 9**] extended this concept to the case of functional differential equations. The existence results in [**4** and **7**], however, are essentially related to perturbations—in some sense—of an ODE. Although more general than the preceding ones, also the existence results in [**2** and **8**] do not seem to be applicable to a scalar equation of the form

$$x'(t) = f(x(t-\tau)), \qquad f\colon \mathbf{R} \to \mathbf{R}, \tag{2}$$

as is shown in §2. This fact raises the problem of finding another definition of a guiding function for (1) in order to obtain further existence results.

In §3 we present a different definition and show how it can be applied to some examples. The new definition takes into account the function x_t and its L^2-norm.

In §4 we then prove an existence theorem for periodic solutions to equation (1) which can also be applied to the examples given in §3.

2. Some remarks on Mawhin's definition of a guiding function. Let $r > 0$ be a fixed number and denote by $\mathscr{C}$ the space of continuous functions from $[-r, 0]$ to $\mathbf{R}^N$, with the usual sup norm. We shall consider the following problem:

$$(\mathrm{P}_1) \qquad \begin{cases} x'(t) = q(t, x_t), \\ x(0) = x(T), \end{cases}$$

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where $T > 0$ is fixed, q: $\mathbf{R} \times \mathscr{C} \to \mathbf{R}^N$ is continuous, transforms bounded sets of $\mathbf{R} \times \mathscr{C}$ into bounded sets of $\mathbf{R}^N$, and is T-periodic with respect to t, i.e.

$$q(t + T, \varphi) = q(t, \varphi) \quad \forall t, \varphi.$$

Whenever x: $\mathbf{R} \to \mathbf{R}^N$ is continuous, x_t is the function of $\mathscr{C}$ defined by

$$x_t(s) = x(t + s) \quad \forall t, s.$$

In [**2**, **8** and **9**], Mawhin introduced the following

DEFINITION 1. A C^1-function V: $\mathbf{R}^N \to \mathbf{R}$ is said to be a guiding function for (P_1) iff there exists $\rho > 0$ such that

$$(\nabla V(x(t)), q(t, x_t)) > 0$$

for every T-periodic continuous function x and every $t \in \mathbf{R}$ for which $|x(t)| \geqslant \rho$ and $|V(x(t))| \geqslant |V(x(s))|$ $\forall s \in [0, T]$.

REMARKS. In the particular case of an ODE, the above definition "includes" the one given by Krasnosel'skii (cf. [**3**]). Moreover, if we consider the following particular FDE,

$$x'(t) = g(t, x(t), x_t) \tag{3}$$

with the usual periodic boundary conditions, a sufficient condition for a C^1-function W to be a guiding function is that there exists $\rho > 0$ such that

$$(\nabla W(x), g(t, x, \phi)) > 0$$

for every $t \in \mathbf{R}$, $\phi \in \mathscr{C}$, and $\|x\| \geqslant \rho$. The latter condition was indeed used in [**7**] by Mawhin to define a guiding function for (3), generalizing a preceding definition of Krasnosel'skii [**4**] in the case of a delay differential equation. In both cases, however, the asymptotically dominating term on the right-hand side of the equation must be the one in $x(\cdot)$, i.e. these definitions work essentially for "perturbations" (in some sense) of an ODE.

The following result [**2**] generalizes a theorem of Krasnosel'skii [**3**].

THEOREM 1. *Let $V_0, \ldots, V_m$ be guiding functions for* (P_1) *such that*
(i) $\lim_{\|x\| \to \infty} \{|V_0(x)| + \cdots + |V_m(x)|\} = \infty$,
(ii) $\deg(\nabla V_0, B_{\rho_0}, 0) \neq 0$.
Then (P_1) *has a solution.*

REMARK. It is easy to see that (ii) is equivalent to

$$\deg(\nabla V_i, B_{\rho_i}, 0) \neq 0 \quad \forall i = 0, \ldots, m,$$

where ρ_i stands for ρ in the definition of V_i.

We shall show now that the above theorem, although quite general, cannot be applied to the scalar equation (2), where $\tau \in (0, T)$. Suppose V_0: $\mathbf{R} \to \mathbf{R}$ is a guiding function for (2). Then in particular for each constant $u \in \mathbf{R}$ with $|u| \geqslant \rho_0$ the following holds:

$$\frac{d}{dx} V_0(u) \cdot f(u) > 0.$$

Moreover, if we assume (ii) of Theorem 1 above, i.e.

$$\deg\left(\frac{d}{dx}V_0,\ ]-\rho_0,\rho_0[,0\right)\neq 0,$$

then both dV_0/dx and f must have constant and equal sign in $[\rho_0,+\infty)$—say positive—*and opposite sign*—say negative—*in* $(-\infty,-\rho_0]$. We claim now that V_0 must be bounded. In fact, suppose for example that $\lim_{x\to+\infty}V_0(x)=+\infty$ (V_0 is increasing in $[\rho_0,+\infty)$). Then take $w\geqslant\rho_0$ such that $V_0(w)=\max_{-\rho_0\leqslant y\leqslant\rho_0}|V_0(y)|$ and let x: $\mathbf{R}\to\mathbf{R}$ be a T-periodic continuous function such that for a $t\in[0,T]$, $x(t)=w$, $x(t-\tau)=-\rho_0$, and moreover $-\rho_0\leqslant x(s)\leqslant w$ $\forall s\in[0,T]$. Then we have

$$\frac{d}{dx}V_0(x(t))\cdot f(x(t-\tau))<0$$

with $|x(t)|\geqslant\rho_0$ and $|V_0(x(t))|\geqslant|V_0(x(s))|$ $\forall s\in[0,T]$, which is in contradiction to the fact that V_0 is a guiding function for (2). Analogously a contradiction arises if we suppose $\lim_{x\to-\infty}V_0(x)=+\infty$. Then V_0 must be bounded and the same is true for any other guiding function V_i, in view of the remark above. In conclusion, condition (i) of the theorem cannot be satisfied.

3. A new definition and some examples. In the following we shall denote by C_T the space of T-periodic continuous functions x: $\mathbf{R}\to\mathbf{R}^N$ with the norm $\|x\|_\infty=\sup_{t\in[0,T]}\|x(t)\|$ and by $\|\cdot\|_2$ the L^2-norm, i.e. $\|x\|_2=(\int_0^T\|x(s)\|^2\,ds)^{1/2}$.

Here is a new definition of a guiding function.

DEFINITION 2. A C^1-function V: $\mathbf{R}^N\to\mathbf{R}$ is said to be an L^2-guiding function for (P_1) iff there exists $R>0$ such that

$$\int_0^T(\nabla V(x(s))\,|\,q(s,x_s))\,ds>0 \tag{4}$$

for every T-periodic C^1-function x with $\|x\|_2\geqslant R$ and $\|x'(t)\|\leqslant\|q(t,x_t)\|$ $(0\leqslant t\leqslant T)$.

This definition has been suggested by the fact, pointed out by Lasota-Yorke [**6**] and Nussbaum [**10**], that an ordinary differential equation in the Hilbert space L^2 can be associated with the functional differential equation we are considering. The Krasnosel'skii definition of a guiding function for the associated ODE was the inspiring point of departure towards the new definition. However our definition does not reduce to Krasnosel'skii's one in the case of an ODE, but they are quite independent. The L^2-norm appearing in the definition is very useful in handling concrete applications, as the following examples show.

EXAMPLE 1. Consider the delay differential problem

$$(\mathrm{P}_2)\qquad\begin{cases}x'(t)=f(t,x(t-\tau)),\\ x(0)=x(T),\end{cases}$$

where f: $\mathbf{R}^{N+1}\to\mathbf{R}^N$ is continuous, T-periodic in t, and there exist positive constants c, $\overline{R}$, M such that

$$(x\,|\,f(t,x))\geqslant c\quad(\|x\|\geqslant\overline{R}),$$
$$\|f(t,x)\|\leqslant M\quad(\text{all }(t,x)\in\mathbf{R}\times\mathbf{R}^N).$$

If $\tau < c \cdot M^{-2}$, then $V(x) = \frac{1}{2}\|x\|^2$ is a L^2-guiding function for (P_2). In fact there certainly exists $R > 0$ such that, for each T-periodic C^1-function x with $\|x\|_2 \geqslant R$ and $\|x'(t)\| \leqslant M$ $(0 \leqslant t \leqslant T)$, it has to be $\|x(t)\| \geqslant \overline{R}$ $\forall t \in [0, T]$. Then we have

$$\begin{aligned}\int_0^T (x(s) \mid f(s, x(s-\tau)))\, ds &= \int_0^T (x(s-\tau) \mid f(s, x(s-\tau)))\, ds \\ &\quad + \int_0^T (x(s) - x(s-\tau) \mid f(s, x(s-\tau)))\, ds \\ &\geqslant [c - \tau M^2]T > 0.\end{aligned}$$

EXAMPLE 2. Let us consider the problem

$$(P_3) \qquad \begin{cases} x'(t) = \nabla F(x(t)) + g(t, x_t), \\ x(0) = x(T), \end{cases}$$

where g is continuous, T-periodic in t, and maps bounded sets into bounded sets. Suppose there exist positive constants m, k, and $\alpha \geqslant 1$ such that

$$(5) \qquad \|\nabla F(x)\| \geqslant m\|x\|^\alpha - k \quad \forall x \in \mathbf{R}^N$$

and

$$\limsup_{\substack{\|x\|_2 \to \infty \\ x \in C_T}} \frac{\|\tilde{g}x\|_2}{\|x\|_2^\alpha} < mT^{(1-\alpha)/2},$$

where $\tilde{g}$ is the Nemitskii operator associated with g:

$$\tilde{g}\colon C_T \to C_T\colon [\tilde{g}(x)](t) = g(t, x_t).$$

We shall see then that F is an L^2-guiding function for (P_3). The embedding $L^{2\alpha} \hookrightarrow L^2$ together with (5) gives us the inequalities

$$\|\nabla F(x(\cdot))\|_2 \geqslant m\|x\|_{2\alpha}^\alpha - k\sqrt{T} \geqslant mT^{(1-\alpha)/2}\|x\|_2^\alpha - k\sqrt{T}.$$

Now, using Hölder and the triangular inequalities, we get

$$\begin{aligned}\int_0^T (\nabla F(x(s)) \mid \nabla F(x(s)) + g(s, x_s))\, ds &\geqslant \|\nabla F(x(\cdot))\|_2 [\|\nabla F(x(\cdot))\|_2 - \|\tilde{g}x\|_2] \\ &\geqslant \|\nabla F(x(\cdot))\|_2 \left[mT^{(1-\alpha)/2} - \frac{\|\tilde{g}x\|_2}{\|x\|_2^\alpha} - k\sqrt{T}\,\frac{1}{\|x\|_2^\alpha} \right] \|x\|_2^\alpha \\ &> 0\end{aligned}$$

if $\|x\|_2$ is large enough.

EXAMPLE 3. Consider the problem

$$(P_4) \qquad \begin{cases} x'(t) = Ax(t) + g(t, x_t), \\ x(0) = x(T), \end{cases}$$

where A is an $N \times N$ matrix and g is continuous, T-periodic in t and maps bounded sets into bounded sets. We consider two cases:

(i) Suppose there exists a positive constant ε such that

$$(6) \qquad (Ax \mid x) \geqslant \varepsilon\|x\|^2 \quad \forall x \in \mathbf{R}^N.$$

Then, if

$$\limsup_{\|x\|_2 \to \infty} \frac{\|\tilde{g}x\|_2}{\|x\|_2} < \varepsilon$$

($\tilde{g}$ as in Example 2), $V(x) = \frac{1}{2}\|x\|^2$ is an L^2-guiding function for (P_4). In fact

$$\int_0^T (\nabla V(x(s)) \mid Ax(s) + g(s, x_s))\, ds \geqslant \varepsilon\|x\|_2^2 - \|x\|_2\|\tilde{g}x\|_2 > 0$$

if $\|x\|_2$ is large enough.

(ii) It can be shown [**3**] that if the matrix A has eigenvalues with both positive and negative real parts, then there exists an $N \times N$ matrix D such that

$$(Dx \mid Ax) \geqslant \|x\|^2 \quad \forall x \in \mathbf{R}^N,$$

Suppose there exist positive constants k_1, k_2 such that

$$\int_0^T (Dx(s) \mid g(x, x_s))\, ds \geqslant (-1 + k_1)\|x\|_2^2 - k_2$$

for every T-periodic C^1-function x. It is easy to see then that

$$V(x) = \tfrac{1}{2}(Dx \mid x)$$

is an L^2-guiding function for (P_4).

4. An existence result. Here is our main result. It is not difficult to see how it can be applied to the above examples.

THEOREM 2. *Consider* (P_1) *and suppose moreover the Nemitzkii operator associated to q transforms L^2-bounded sets into L^2-bounded sets. If there exists an L^2-guiding function V for* (P_1) *such that*

$$\deg(\nabla V, B_R, 0) \neq 0, \tag{7}$$

then (P_1) *has a solution $x \in C_T$.*

REMARK. Condition (7) is satisfied for instance if V is even or if $\lim_{\|x\| \to \infty} V(x) = \pm\infty$.

PROOF. We shall apply a theorem of Mawhin (cf. [**8**, Theorem IV.13]). Define the following operators

$$L\colon \operatorname{dom} L \equiv \{x \in C_T \mid x \in C^1\} \to C_T, \qquad Lx = x',$$
$$N\colon C_T \to C_T, \qquad (Nx)(t) = q(t, x_t).$$

Standard considerations (see, e.g., [**8**]) show that L is a Fredholm mapping of index 0, and N (the Nemitzkii operator) is L-completely continuous. Moreover, $\ker L = \mathbf{R}^N$ and we define the following projector operator:

$$Q\colon C_T \to \mathbf{R}^N : Qx = \frac{1}{T}\int_0^T x(s)\, ds.$$

Fix $\lambda \in (0, 1)$ and let $x \in \operatorname{dom} L$ be a solution to the equation $Lx = \lambda Nx$ which is indeed equivalent to the problem

$$(P_\lambda) \qquad \begin{cases} x'(t) = \lambda q(t, x_t), \\ x(0) = x(T). \end{cases}$$

Clearly, V is an L^2-guiding function for (P_λ), too. On the other hand we have

$$\int_0^T (\nabla V(x(s)) \mid q(s, x_s))\, ds = \frac{1}{\lambda}\int_0^T (\nabla V(x(s)) \mid x'(s))\, ds$$
$$= \frac{1}{\lambda}\int_0^T \frac{d}{ds} V(x(s))\, ds = \frac{1}{\lambda}[V(x(T)) - V(x(0))] = 0$$

which implies

$$\|x\|_2 < R. \tag{8}$$

Moreover, the hypothesis on N gives us

$$\|x'\|_2 < C \tag{9}$$

for some $C > 0$. It then follows from (8) and (9) that

$$\|x\|_\infty < C'. \tag{10}$$

Now, define $\Omega = \{x \in C_T\colon \|x\|_\infty < r\}$, where $r = \max\{R, C', R \cdot T^{-1/2}\}$. Estimate (10) assures us that

$$Lx \neq \lambda Nx \quad \forall x \in \partial\Omega.$$

The definition of an L^2-guiding function gives us

$$\int_0^T (\nabla V(u) \mid q(s, u))\, ds > 0 \tag{11}$$

for every constant $u \in \mathbf{R}^N$ with $\|u\| \geqslant R \cdot T^{-1/2}$, and hence

$$QNx \neq 0 \quad \forall x \in \ker L \cap \partial\Omega.$$

Finally, (11) together with (7) gives us

$$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) = \deg(\nabla V, B_r, 0) \neq 0$$

an thus Theorem IV.13 in **[8]** can be applied to complete the proof.

In the case of ordinary differential equations, the above theorem together with standard differential inequalities yield the following known result (see e.g. **[1, 5]**):

COROLLARY. *Consider the problem*

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x(T), \end{cases} \tag{P_5}$$

where $f\colon \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous and T-periodic in t. If there exist a C^1-function $W\colon \mathbf{R}^N \to \mathbf{R}$ and $\tilde{R} > 0$ such that

$$(\nabla W(u) \mid f(t, u)) > 0 \qquad (t \in [0, T], \|u\| \geqslant \tilde{R}) \tag{12}$$

and

$$\deg(\nabla W, B_{\tilde{R}}, 0) \neq 0,$$

and if moreover f is such that

$$\|f(t, u)\| \leqslant K_1\|u\| + K_2$$

for some positive constants K_1 and K_2, then W is an L^2-guiding function for (P_5) and (P_5) has a solution.

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