ON PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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ABSTRACT. Necessary and sufficient conditions for existence of periodic solutions of differential equations containing singularities are given. Our theorems apply to \( u'' + 1/u^\alpha = h(t) \) for all \( \alpha > 0 \) and to \( u'' - 1/u^\alpha = h(t) \) if \( \alpha \geq 1 \), and for this case \( \alpha \geq 1 \) is an essential condition.

1. In this note we consider a class of second order scalar differential equations with periodic forcing, zero damping, and a restoring force which becomes infinite at a finite displacement, which we take to be zero. We give a necessary and sufficient condition for the existence of a periodic solution for equations in this class. Our first theorem will show that if \( h(t) \) is continuous and \( T \)-periodic, then for all \( \alpha > 0 \) there exists a positive \( T \)-periodic solution of

\[ u''(t) + 1/u(t)^\alpha = h(t) \]

if and only if \( h(t) \) has a positive mean value. Our second theorem will show that if \( \alpha > 1 \), then the repulsive type equation

\[ u''(t) - 1/u(t)^\alpha = h(t) \]

has a positive \( T \)-periodic solution if and only if \( h(t) \) has a negative mean value.

In the last section, we show that this result is the best possible, by showing that for any \( \alpha \), \( 0 < \alpha < 1 \), we can choose \( h \) so that \( h \) has negative mean value and the equation has no \( T \)-periodic solution.

Our methods consist of sub- and super-solution arguments and truncation arguments based on a priori upper and lower bounds of periodic solutions which permit reduction to the case of bounded nonlinearities and the application of the results in [3] (see also [1, p. 121 or 4, p. 23]).

2. In this section we consider a general class of problems which includes (1.1) with \( \alpha > 0 \).

THEOREM 2.1. Let \( g \) be a real valued continuous function defined on \((-\infty, 0) \cup (0, \infty)\) such that \( g(\xi) \to 0 \) as \( |\xi| \to \infty \), \( g(\xi) \to +\infty \) as \( \xi \to 0^+ \), \( g(\xi) \to -\infty \) as \( \xi \to 0^- \), and \( g(\xi)\xi > 0 \) for \( \xi \neq 0 \). Let \( h(t) \) be defined and continuous for \( -\infty < t < \infty \) and satisfy \( h(t) \equiv h(t + T) \) for some \( T > 0 \). A necessary and sufficient condition that there exists a \( T \)-periodic solution of

\[ u'''(t) + g(u(t)) = h(t) \]

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is that
\[ \int_0^T h(s) \, ds \neq 0 \]  

Before proving the theorem let us make clear that by a solution of (2.2) we mean a $C^2$-function which satisfies the differential equation.

**Proof.** Suppose $u(t)$ is a $T$-periodic solution of (2.2). Since $u(t) \neq 0$ for all $t$ the assumptions on $g$ imply that $g(u(t))$ is either always positive or always negative. Integrating both sides of (2.2) from $t = 0$ to $t = T$ we obtain
\[ \int_0^T h(t) \, dt = \int_0^T g(u(t)) \, dt \neq 0. \]

Therefore (2.3) is necessary for the existence of a $T$-periodic solution.

Conversely, suppose that (2.3) holds. We shall consider only the case
\[ \int_0^T h(t) \, dt = \int_0^T g(u(t)) \, dt \geq 0 \]

and show that this implies the existence of a positive $T$-periodic solution of (2.2).

The proof that $h_0 < 0$ implies the existence of a negative $T$-periodic solution of (2.2) is similar.

If $\varepsilon > 0$ is chosen so small that $g(\varepsilon) - h(t) > 0$ for all $t$, then the constant function $u_\varepsilon(t) \equiv \varepsilon$ is a *sub-solution* of the boundary value problem given by equation (2.2) and $T$-periodic boundary conditions since
\[ u_\varepsilon''(t) + g(u_\varepsilon(t)) \geq h(t). \]

(See, for example, [2] for a discussion of the method of sub- and super-solutions applied to problems with periodic boundary conditions.) To prove the existence of a positive periodic solution of (2.2) it is only necessary to find a $T$-periodic $C^2$ function $u^*(t)$ such that
\[ u^*''(t) + g(u^*(t)) \leq h(t) \]
and $u_\varepsilon(t) < u^*(t)$ for all $T$. ($u^*(t)$ is a *super-solution.*)

Since the continuous $T$-periodic function $h(t) - h_0$ has mean value zero, there exists a $C^2$-function $w(t)$ which is $T$-periodic such that $w''(t) = h(t) - h_0$. Using the fact that $g(\xi) \to 0$ as $\xi \to +\infty$ we may choose a constant $c > 0$ so large that $u^*(t) \equiv c + w(t) > u_\varepsilon(t)$ for all $t$ and $g(u^*(t)) < h_0$ for all $t$. It then follows that (2.6) holds and by earlier remarks this proves the existence of a $T$-periodic solution $u(t)$ of (2.2).

3. In this section we consider a class of problems which includes (1.2) if $\alpha > 1$. Here we use a truncation argument which also applies to equations of the type (1.1), provided $\alpha > 1$, and which gives the existence of a $T$-periodic $C^1$-function $u$ which is strictly positive everywhere and which solves the equation in a weak sense for every $h \in L^1(0, T)$. Analogously one can get negative solutions of (1.2). We use $\| \cdot \|_p$ to indicate the $L^p$ norm on $[0, T]$. 

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PROPOSITION 3.1. Let $u$ be a $T$-periodic distribution and let $u'' \in L^1(0,T)$. Then if one indicates by $(u'')^+$ (resp. $(u'')^-$) the positive (resp. negative) part of $u''$, then for both the + and the − sign one has

$$\|u'\|_\infty < \|(u'')^\pm\|_1. \tag{3.2}$$

PROOF. Let $t_0$ be a point of minimum for $u$. Therefore, since $u \in C^1$, one has

$$u'(t_0) = u'(t_0 + T) = 0. \tag{3.3}$$

We fix $t \in [t_0, t_0 + T]$ and obtain

$$u'(t) = \int_{t_0}^{t_0 + T} u''(s) \, ds \leq \int_{t_0}^{t_0 + T} (u''(s))^+ \, ds = \|(u'')^+\|_1, \tag{3.4}$$

and analogously

$$u'(t) = \int_{t}^{t_0 + T} (u''(s))^+ \, ds \geq -\int_{t}^{t_0 + T} (u''(s))^+ \, ds = -\|(u'')^+\|_1. \tag{3.5}$$

(3.4) and (3.5) give the estimate (3.2) for the + sign. Interchanging $u$ with $-u$, one proves (3.2) with the − sign. □

The preceding result gives an a priori bound from below for any classical positive $T$-periodic solution of

$$u'' - g(u) = h(t), \tag{3.6}$$

where $g$ is a positive function defined on $(0, +\infty)$ such that

$$\lim_{s \to 0^+} g(s) = +\infty, \quad \int_0^1 g(x) \, dx = +\infty. \tag{g_1}$$

LEMMA 3.7. Let (g_1) hold. Then for any constant $M > 0$ there exists a constant $\varepsilon > 0$ such that for any $T$-periodic continuous function $h$ such that $\|h\|_1 < M$ and any $T$-periodic positive classical solution $u$ of (3.6) one has

$$\forall t \in \mathbb{R} : u(t) > \varepsilon. \tag{3.8}$$

PROOF. Let $\xi \in \mathbb{R}_+$ be such that

$$\forall x \leq \xi : g(x) > T^{-1}M. \tag{3.9}$$

If one integrates both sides of (3.6) one has

$$\int_0^T g(u(t)) \, dt = \int_0^T -h(t) \, dt \leq M, \tag{3.10}$$

and therefore by (3.9) one sees that there exists $t_1 \in \mathbb{R}$ such that $u(t_1) > \xi$. Now fix $\varepsilon > 0$ in such a way that

$$\int_\varepsilon^\xi g(x) \, dx > 2M^2. \tag{3.11}$$
Multiplying both sides of \((3.6)\) by \(u'\) and integrating between \(t_1\) and \(t\) we get
\[
\int_{t_1}^{t} u''(s)u'(s)\,ds - \int_{t_1}^{t} g(u(s))u'(s)\,ds = \int_{t_1}^{t} h(s)u'(s)\,ds.
\]
Since \(\|u'\|_{\infty} \leq \|(g(u) + h)\|_{1} \leq \|h\|_{1} \leq M\) because \(g(u) \geq 0\) and from \((3.2)\), we see that for \(t \in [t_1, t_1 + T_0]\)
\[
\frac{1}{2}(u'(t))^2 - \frac{1}{2}(u'(t_1))^2 - \int_{t_1}^{u(t)} g(x)\,dx \leq M^2
\]
and finally
\[
\int_{u(t_1)}^{u(t)} g(x)\,dx \leq M^2 + \frac{(u'(t_1))^2}{2} \leq 2M^2.
\]
Since \(u(t_1) > \xi\) and \(g\) is positive, from \((3.11)\) one gets \(u(t) > \varepsilon\). □

REMARK. The assumption \((g_1)\) has been used in the previous lemma only in order to determine \(\xi\) and \(\varepsilon\) in such a way that \((3.9)\) and \((3.11)\) hold. Therefore the estimate is verified only provided \((3.9)\) and \((3.11)\) are true, and \(g\) could be defined in all \(\mathbb{R}\). Moreover only \((3.9)\) is affected by the values taken by \(g\) at the left side of \(\varepsilon\). This last observation is the point on which the truncation used in the following theorem is based.

**THEOREM 3.12.** Let \(h \in L^1(0,T)\) be given and assume \(h\) to be \(T\)-periodic in \(\mathbb{R}\). Suppose that \((g_1)\) holds and that \(g > 0\) and
\[
\lim_{x \to +\infty} g(x) = 0.
\]
Then \((3.6)\) has a \(T\)-periodic weak solution iff \(\int_0^T h(t)\,dt < 0\).

PROOF. The necessity comes immediately from \((3.10)\) since \(g\) is positive. For the sufficiency, first assume \(h\) continuous and let \(M > \|h\|_1\). We fix \(\xi, \varepsilon\) in such a way that \((3.9)\) and \((3.11)\) hold. Then put
\[
\bar{g}(s) = \begin{cases} \frac{2}{T} \int_0^T h(t)\,dt & \text{if } s > \xi, \\ g(\xi) & \text{if } s \leq \xi, \end{cases}
\]
defining in this way \(\bar{g}\) on \(\mathbb{R}\). Since \((3.9)\) is of course preserved if we change \(g\) with \(\bar{g}\) then by Lemma 3.7 and the subsequent remark we know that the solutions of
\[
(u'') - \bar{g}(u) = h
\]
are bounded below by \(\varepsilon\) and therefore they are precisely the positive solutions of \((3.6)\). The resonance theorem proved in \([3]\) states that \((3.14)\) has a \(T\)-periodic solution provided
\[
-g(\varepsilon) = \lim_{x \to +\infty} -\bar{g}(x) < \frac{1}{T} \int_0^T h(x)\,dx < \lim_{x \to +\infty} -\bar{g}(x) = 0
\]
and this last condition is verified by the assumptions which we are making and by \((3.9)\). Finally if \(h\) is not continuous let \((h_n)_n\) be a sequence of continuous functions which converges to \(h\) in \(L^1(0,T)\), and let \(M > \|h_n\|_1 \forall n \in \mathbb{N}\). Then find \(\xi\) and \(\varepsilon\) according to \((3.9)-(3.11)\) and define the truncation \(\bar{g}\). For any \(n\) the first part
of the theorem provides a solution \( u_n \) of \( u'' - \bar{g}(u) = h_n \), such that \( u_n > \varepsilon \). By standard arguments one can pass to the limit and get a solution of (3.6).

Let us remark that Proposition 3.1 is also useful in order to solve other kinds of periodic equations with positive nonlinearities. If for instance \( f \) is positive and \( \lim_{s \to +\infty} f(x) = 0, \lim_{s \to -\infty} f(s) = +\infty \) (e.g. \( f(s) = e^{-s} \)), given any \( \xi_1 \in \mathbb{R} \) such that one has \( f(\xi) > T^{-1}\|h\|_1 \) for a given \( h \in L^1(0,T) \) and \( \xi < \xi_1 \), then any positive solution of

\[
(3.15) \quad u'' - f(u) = h
\]

verifies the estimate \( \xi_1 - T\|h\|_1 \leq u(t) \) for all \( t \) in \( \mathbb{R} \).

The same estimate holds if \( f \) is replaced by \( \bar{f} \), where

\[
\bar{f}(s) = \begin{cases} f(\xi_1 - T\|h\|_1) & \text{if } s \leq \xi_1 - T\|h\|_1, \\ f(s) & \text{if } \xi_1 - T\|h\|_1 < s. \end{cases}
\]

Replacing \( f \) by \( \bar{f} \), by the results in [3], one can solve (3.15) if

\[
\lim_{s \to -\infty} -\bar{f}(s) = -f(\xi_1 - T\|h\|_1) < \frac{1}{T} \int_0^T h < \lim_{s \to +\infty} -\bar{f}(s) = 0.
\]

This last condition is verified provided \( \int_0^T h < 0 \). Of course this condition is necessary if \( f \) is strictly positive everywhere.

4. In this last section we show that the result in the previous theorem cannot be extended to (1.2) for \( 0 < \alpha < 1 \). More generally we assume that \( g \) is strictly positive and that

\[
(3.72) \quad \lim_{s \to 0^+} g(s) = +\infty, \quad \lim_{s \to +\infty} g(s) = 0, \quad \int_0^1 g(s) \, ds < +\infty.
\]

If \( g \) verifies (3.72) we have

**Theorem 4.1.** \( \forall T > 0 \exists M_0 > 0 \) such that \( \forall M > M_0 \exists h, \) a continuous \( T \)-periodic negative function, such that (3.6) has no solution and \( -\int_0^T h = M \).

**Proof.** For simplicity we let \( h \) be a step function; a small regularization of \( h \) does not affect the computations below. We take \( h = -\varepsilon^{-1}M \chi_{[t_1, t_1 + \varepsilon]} \), where \( \varepsilon \) is a small positive real number and \( \chi_{[t_1, t_1 + \varepsilon]} \) denotes the characteristic function of the interval \( [t_1, t_1 + \varepsilon] \) for \( t_1 \in \mathbb{R} \). Suppose \( u \) solves (3.6) and fix \( \xi \in \mathbb{R} \) such that (3.9) holds. As in Lemma 3.7 we see, using (3.10), that \( \max u > \xi \). Of course \( \xi \) depends only on \( M \). By the result in Proposition 3.1, \( |u'| \) is bounded by \( M \). Since \( h = 0 \) in \( [0,T] \setminus [t_1, t_1 + \varepsilon] \), we have that in this interval \( u'' = g(u) + h = g(u) > 0 \). Thus, the point of \( \max u \) must belong to \( [t_1, t_1 + \varepsilon] \). Collecting all this information we have

\[
\inf_{[t_1, t_1 + \varepsilon]} u \geq \xi - \varepsilon M > \xi/2,
\]

provided we choose \( \varepsilon < M^{-1}\xi/2 \). We also choose \( \varepsilon \) so small that \( \varepsilon \max_{s \geq \xi/2} g(s) < M/2 \). We have

\[
(4.2) \quad u'(t_1 + \varepsilon) - u'(t_1) = \int_{t_1}^{t_1 + \varepsilon} u''(s) \, ds = \int_{t_1}^{t_1 + \varepsilon} h(s) \, ds + \int_{t_1}^{t_1 + \varepsilon} g(s) \, ds < -\frac{M}{2}.
\]
Therefore $|u'(t)| > M/4$ for $t = t_1$ or for $t = t_1 + \varepsilon$. Assume that $|u'(t_1 + \varepsilon)| > M/4$ and for simplicity of notation let $t_1 = 0$.

We also observe that if we fix $\xi'$ such that $\sup_{s > \xi'} g(s) < T^{-1}M$, by Proposition 3.1 and by (3.10) we have $\sup u < \xi' + TM$, and $\xi'$ also does not depend on $M \geq \overline{M}$. Let $t_0$ be a point of min $u$ on $[0, T]$. In $[0, \varepsilon]$

$$u'' = g(u) + h < \varepsilon^{-1}M/2 - \varepsilon^{-1}M < 0,$$

so $t_0 \in [\varepsilon, T]$, and therefore $h = 0$ in $[\varepsilon, t_0]$. Multiplying (3.6) by $u'$ and integrating between $\varepsilon$ and $t_0$ we get

$$u'(\varepsilon)^2 - u'(t_0)^2 = 2 \int_{u(t_0)}^{u(\varepsilon)} g(s) \, ds \leq 2 \int_0^{\varepsilon' + TM} g(s) \, ds,$$

which implies

$$\int_0^{\varepsilon' + TM} g(s) \, ds \geq \frac{M^2}{32}.$$ (4.3)

The assumption $(g_2)$ clearly implies that the left-hand side of (4.3) is a sublinear function of $M$, so (4.3) is definitely false for $M$ large and the theorem is proved.

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