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ON UNILATERAL AND BILATERAL nTH PEANO DERIVATIVES

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ABSTRACT. A valid proof is given of the assertion that an nth Peano derivative that is allowed to attain infinite values is still a function of Baire class one. Also, it is shown that a finite, unilateral nth Peano derivative is a function of Baire class one. Finally, an example is given that if infinite values are allowed (actually just $+\infty$) a unilateral nth Peano derivative need not be of Baire class one.

1. Introduction. It is obvious that if f has a finite derivative, $f'(x)$, everywhere, then f' is of Baire class one because in this case f is continuous. However, if f' is allowed to attain infinite values, then f' need not be continuous and, consequently, it is no longer trivial that f' is a Baire one function. Nevertheless, even in this more general case f' is a Baire one function, as was established by Zahorski in 1950. (See the corollary on p. 15 of [**12**].) That a finite nth Peano derivative, f_n, which exists everywhere is of the first Baire class was first proved by Denjoy. (See [**4**, Theorem IV, p. 289].) (For a more extensive discussion of the nth Peano derivative, see [**5**].) In [**1**] it is asserted even if f_n is allowed to attain infinite values, it is still a function of Baire class one. Unfortunately, as was exhibited in [**5**], there is an error in the proof. Here we offer a correct proof of that assertion and go on to show that in this case when f_n is bounded above or below on an interval, $f_n = f^{(n)}$ on that interval. ($f^{(n)}$ denotes the ordinary nth derivative of f.) We then deduce that f_n has the Denjoy-Clarkson property (or Zahorski's property M_2) and, if $n \geq 2$, that f_n has the Darboux property.

Turning our attention to the unilateral case we show that if, say, the right nth Peano derivative, $f_{n+}(x)$ exists finitely everywhere, then f_{n+} is Baire one. The proof relies on showing that under these conditions every nonempty, perfect set contains a portion relative to which $f, f_{1+}, \dots, f_{(n-1)+}$ are continuous. This property is referred to both as generalized continuous and B_1^*. In contrast to the case of bilateral nth Peano derivatives, if f_{n+} is allowed infinite values, then f_{n+} need not be Baire class one. We give an example.

2. Definitions and notation. In this section we set out the notation to be used and recall for the reader the definitions of the concepts dealt with in this paper. The real line will be denoted by R and, for $E \subset R$, $\operatorname{cl} E$ is the closure of E.

DEFINITION. Let f be an extended real-valued function; that is, the range of f is $R \cup \{-\infty, +\infty\}$. Then f is said to have the Denjoy-Clarkson property if, for each pair of extended real numbers $a < b$, $f^{-1}(a, b)$ either is empty or has positive (Lebesgue) measure.

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Let n be a positive integer and recall that a real-valued function f is said to have a finite nth Peano derivative at x if there are n numbers $f_1(x), \dots, f_n(x)$ and a function $\varepsilon_n(x,h)$ defined for h in some open interval about 0 such that

$$f(x+h) = f(x) + hf_1(x) + h^2 f_2(x)/2 + \cdots + h^{n-1} f_{n-1}(x)/(n-1)! + h^n (f_n(x) + \varepsilon_n(x,h))/n!$$

and

$$\lim_{h\to 0} \varepsilon_n(x,h) = 0.$$

For each $j = 1, 2, \dots, n$, the number $f_j(x)$ is called the jth Peano derivative of f at x. The reader unfamiliar with the nth Peano derivative should consult the survey article [**5**] for facts and references. For the sake of motivation we confine ourselves to the simple observation that if f has a finite nth Peano derivative, then

$$f_n(x) = \lim_{h\to 0} \frac{f(x+h) - f(x) - hf_1(x) - \cdots - (h^{n-1}/(n-1)!) f_{n-1}(x)}{h^n/n!}.$$

DEFINITION. If f has a finite $(n-1)$th Peano derivative at x, then we will say $f_n(x) = +\infty$ $(-\infty)$ provided

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - hf_1(x) - \cdots - (h^{n-1}/(n-1)!) f_{n-1}(x)}{h^n/n!} = +\infty \ (-\infty).$$

The above definitions of $f_n(x)$ can be altered to define the right nth Peano derivative, $f_{n+}(x)$, simply by replacing $\lim_{h\to 0}$ with $\lim_{h\to 0^+}$.

3. Bilateral Peano derivatives. In this the main part of the paper we show that if f has an nth Peano derivative, $f_n(x)$, on a closed set allowing infinite values, then f_n is in Baire one. With the aid of an auxiliary theorem it is concluded that, when the closed set is an interval, f_n has the Denjoy-Clarkson property and $(n \geq 2)$ the Darboux property. The first of these is expected since first (possibly infinite) derivatives have this property. However, only finite first derivatives are Darboux, which could easily lead to the conclusion that the Darboux property is a property of finite derivatives. So that f_n has the Darboux property for $n \geq 2$ is somewhat surprising.

3.1. THEOREM. *Let n be an integer, $n \geq 2$, and let $F \subset R$ be closed. Suppose that for each $x \in F$, $f_n(x)$ exists with infinite values allowed. Then f_n is a function of Baire class one.*

PROOF. Let P be a nonempty, perfect subset of F. It suffices to show that there is a portion of P relative to which f_n is Baire one. For notational purposes let

$$\phi(x,h) = \frac{(f(x+h) - f(x) - hf_1(x) - \cdots - h^{n-1} f_{n-1}(x)/(n-1)!)}{h^n/n!}.$$

Then $f_n(x) = \lim_{h\to 0} \phi(x,h)$. Next set $\psi(x,h) = (\phi(x,h) + \phi(x,-h))/2$. Then again $f_n(x) = \lim_{h\to 0} \psi(x,h)$, but the important observation to make is that, whether n is odd or even, $\psi(x,h)$ involves only the functions $f, f_1, \dots, f_{n-2}$ and, in particular, it does not involve f_{n-1}. (The case $n = 2$ is actually completed by this remark.) By Theorem II, p. 283 of [**4**] there is a portion, Q, of P relative to which $f_1, \dots, f_{n-2}$ are all continuous. Since $n \geq 2$, f is everywhere continuous.

It follows that for $h \neq 0$ fixed, $\psi(x, h)$ is a continuous function of x relative to Q. Letting h tend to 0 through a sequence implies that f_n is Baire one relative to Q which completes the proof.

The next theorem is the auxiliary result mentioned at the beginning of this section.

3.2. THEOREM. *If f_n exists and is bounded above or below on an interval, I, then $f_n = f^{(n)}$, the ordinary nth derivative of f, on I.*

For the case where f_n is finite, the result was proved by Corominas [**3**], Oliver [**8**] and most recently by Verblunsky [**10**]. One can easily check that Verblunsky's proof (pp. 314–318 of [**10**]) is valid in the infinite case as well. The only extension to his argument that must be made is the infinite case of his Lemma 1.

3.3. LEMMA. *Let f_1 be increasing on $[a, b]$ and, if $n > 2$, suppose in addition that $f_1(a) = f_2(a) = \cdots = f_{n-1}(a) = 0$. If $f_n(a) > 0$, then $(f_1)_{n-1}(a) = f_n(a)$.*

PROOF. We need only consider the case $f_n(a) = +\infty$ as the finite case is established by Verblunsky. In this case we have

$$\begin{aligned} +\infty = f_n(a) &= \lim_{h \to 0^+} \frac{f(a+h) - f(a) - hf_1(a)}{h^n/n!} \\ &= \lim_{h \to 0^+} \frac{\int_0^h (f_1(a+t) - f_1(a))\,dt}{h^n/n!}. \end{aligned}$$

Since f_1 is increasing,

$$\int_0^h (f_1(a+t) - f_1(a))\,dt \leq h(f_1(a+h) - f_1(a)).$$

Thus

$$+\infty = \lim_{h \to 0^+} \frac{(f_1(a+h) - f_1(a))}{h^{n-1}/n!} = \lim_{h \to 0^+} \frac{(f_1(a+h) - f_1(a))}{h^{n-1}/(n-1)!}.$$

Therefore, $(f_1)_{n-1}(a) = +\infty$.

3.4. THEOREM. *Let n be an integer, $n \geq 2$, and suppose $f_n(x)$ exists for all x with infinite values allowed. Then*

(1) f_n has the Darboux property and

(2) f_n has the Denjoy-Clarkson property.

PROOF. (1) By Theorem 1, p. 839 of [**7**] it suffices to show that for every real number α the sets $\{x \colon f_n(x) \geq \alpha\}$ and $\{x \colon f_n(x) \leq \alpha\}$ contain the endpoints of any interval they contain. (This result is stated only for real-valued functions, but it is easily seen to hold for extended real-valued functions as well.) In our case it suffices to assume that $f_n(x) \geq 0$ for all $x \in (a, b)$ and to conclude that $f_n(a) \geq 0$ and $f_n(b) \geq 0$. Clearly, f_n is bounded below on $[a, b]$. Consequently, $f_n = f^{(n)}$ on $[a, b]$. Moreover, it follows from Theorem 1(i), p. 14 of [**10**] that since $n \geq 2$, f_{n-1} is continuous on $[a, b]$. Thus on $[a, b]$, f_n is the derivative of the continuous function f_{n-1}. It is well known that the derivative of a continuous function has a Darboux property. So f_n has the Darboux property on $[a, b]$. Consequently, $f_n(a) \geq 0$ and $f_n(b) \geq 0$.

(2) According to Theorem 1, p. 365 of [**11**] a Baire one function has the Denjoy-Clarkson property if it has that property on each interval on which it is bounded above or below. (Again this assertion is proved only for real-valued functions, but a perusal of the proof will show that it works just as well for extended real-valued functions.) If f_n is bounded above or below on an interval, then $f_n = f^{(n)}$ on that interval and in [**2**] it is shown that an ordinary derivative has the Denjoy-Clarkson property.

We conclude this section by noting that it is the Darboux property that necessitates the assumption $n \geq 2$. Assertion (1) is true for $n = 1$.

4. Unilateral nth Peano derivatives. In this section we establish the assertion concerning unilateral nth Peano derivatives made in the introduction. That finite unilateral nth Peano derivatives are of Baire class one can be proved in a manner analogous to the proof given by Denjoy [**4**] that finite (bilateral) nth Peano derivatives are Baire one. As that proof is distributed in the first few sections of Denjoy's article, we outline the procedure here leaving some of the details for the dedicated reader. The first step is a lemma which is of interest in its own right.

4.1. LEMMA. *Let n be a positive integer and let $F \subset R$ be closed. Suppose, for each $x \in F$, $f_{n+}(x)$ exists and is finite. Then each nonempty, perfect subset of F contains a portion relative to which $f, f_{1+}, \dots, f_{(n-1)+}$ are continuous.*

PROOF. Let $x \in F$. Since $f_{n+}(x)$ exists and is finite,

$$f(x+h) = f(x) + hf_{1+}(x) + \cdots + \frac{h^n}{h!} f_{n+}(x) + \frac{h^n}{h!} \varepsilon_n(x, h) \tag{1}$$

where

$$\lim_{h \to 0^+} \varepsilon_n(x, h) = 0.$$

Let $\phi \neq P \subset F$ be perfect, and for each positive integer N let $E_N = \{x \in P : 0 < h < 1/N \text{ implies } |\varepsilon_n(x, h)| < 1\}$. Since $\bigcup_{N=1}^{\infty} E_N = P$, it suffices to show that $f, f_{1+}, \dots, f_{(n-1)+}$ are continuous relative to $\operatorname{cl} E_N$.

Let $x_0 \in \operatorname{cl} E_N$, and let $0 < h_0 < \cdots < h_n < 1/N$. For each $x \in E_N$ with $|x - x_0| < \min\{h_0, 1/N - h_n\}$, $x < x_0 + h_0 < x_0 + h_n < x + 1/N$. For each $i = 0, 1, \dots, n$, let $k_i = x_0 + h_i - x$. Then for $i = 0, 1, \dots, n$,

$$f(x_0 + h_i) - \frac{k_i^n}{n!} \varepsilon_n(x, k_i) = f(x) + k_i f_{1+}(x) + \cdots + \frac{k_i^n}{n!} f_{n+}(x).$$

This can be interpreted as a system of $(n+1)$ linear equations in the unknowns $f(x), f_{1+}(x), \dots, f_{n+}(x)$. This system has a unique solution. For each unknown the solution is expressed as the quotient of two determinants. The entries in the determinant in the denominator involve only the numbers k_i and hence the denominator is a continuous function of x since each k_i is a continuous function of x. The determinant in the numerator involves, besides the numbers k_i, also the numbers $\varepsilon(x, k_i)$ and hence is only bounded for $x \in E_N$. By compactness any sequence in E_N can be thinned down to a subsequence $\{x_j\}$ such that $\{f(x_j)\}$, $\{f_{1+}(x_j)\}, \dots, \{f_{n+}(x_j)\}$ converge to say $a_0, a_1, \dots, a_n$, respectively. For each $h \in (0, 1/N)$ such that f is continuous at $x_0 + h$ (since f is right continuous, f is of Baire class one and hence the set of possible choices for h is a dense G_δ subset of $(0, 1/N)$), using (1) with x

replaced by x_j and letting j tend to ∞ shows that

$$f(x_0+h) = a_0 + ha_1 + \cdots + \frac{h^n}{n!}a_n + \frac{h^n}{n!}\lim_{j\to\infty}\varepsilon_n(x_j,h)$$

where $|\lim_{j\to\infty}\varepsilon_n(x_j,h)| \le 1$. It follows that $a_0 = f(x_0)$, $a_1 = f_{1+}(x_0), \ldots, a_{n-1} = f_{(n-1)+}(x_0)$. Since these equalities hold for each sequence, it can be concluded that $\lim_{x\to x_0, x\in E_N} f_{i+}(x) = f_{i+}(x_0)$ for $i = 0, 1, \ldots, n-1$ (where $f_{0+} = f$). It now follows easily that $f, f_{1+}, \ldots, f_{(n-1)+}$ are continuous on $\mathrm{cl}\, E_N$ relative to $\mathrm{cl}\, E_N$.

4.2. THEOREM. *Let n, F and f be as in Lemma* 4.1. *Then f_{n+} is a function of Baire class one on F.*

PROOF. Let $\phi \ne P \subset F$ be perfect. It suffices to show that f_{n+} has a point of continuity in P relative to P. By Lemma 4.1 there is a portion Q of P such that $f, f_{1+}, \ldots, f_{(n-1)+}$ are continuous on Q relative to Q. Let

$$\Phi(x,h) = \frac{f(x+h) - f(x) - hf_{1+}(x) - \cdots - (h^{n-1}/(n-1)!)f_{(n-1)+}(x)}{h^n/n!}.$$

Then $f_{n+}(x) = \lim_{h\to 0^+}\Phi(x,h)$. Let $c \in \mathbf{R}$. Then it is easy to see that

$$\begin{aligned} E_c &= \{x \in Q\colon f_{n+}(x) \ge c\} \\ &= \bigcap_{k=1}^{\infty}\{x \in Q\colon \text{there is an } h \in (0, 1/k) \text{ such that } \Phi(x,h) > c - 1/k\}. \end{aligned}$$

So to show that E_c is a G_δ set it suffices to show that

$$G = \{x \in Q : \text{there is an } h \in (0,1/k) \text{ such that } \Phi(x,h) > c\}$$

is open. Let $x_0 \in G$. There is an $h_0 \in (0, 1/k)$ such that $\Phi(x_0, h_0) > c$. There is a $0 < \delta < \min\{h_0, 1/k - h_0\}$ such that $|x - x_0| < \delta$, $x \in Q$, and $|h - h_0| < \delta$ imply

$$\frac{f(x_0+h_0) - f(x) - hf_{1+}(x) - \cdots - (h^{n-1}/(n-1)!)f_{(n-1)+}(x)}{h^n/n!} > c.$$

Now if $|x - x_0| < \delta$, then there is an h in $(0, 1/k)$ with $|h - h_0| < \delta$ such that $x_0 + h_0 = x + h$. So clearly $\Phi(x,h) > c$. Thus G is open. As a consequence E_c is a G_δ set. By a similar argument $\{x \in Q\colon f_{n+}(x) \le c\}$ is a G_δ set. Thus f_{n+} restricted to Q is a function of Baire class one. So there is an x in Q where f_{n+} restricted to Q is continuous. But since Q is a portion of P, x is a point of continuity of f_{n+} relative to P.

We conclude with an example which can be used to prove that a unilateral nth derivative which is permitted to attain infinite values need not be of Baire class one.

4.3. EXAMPLE. There is a function g defined on $[0,1]$ such that

(a) g is bounded and approximately continuous,

(b) $g'_+(x)$ exists for every x in $[0,1]$ allowing infinite values,

(c) g'_+ is not of Baire class one on $[0,1]$.

PROOF. Let P be a nonempty, perfect subset of $[0,1]$ having measure zero. There is a function h defined on $[0,1]$ such that $h(0) = 0$, $h(1) = 1$, h is nondecreasing on $[0,1]$, and $h'(x) = \infty$ for all x in P. Let $\{(a_n, b_n)\}$ be the intervals contiguous to P. For $x \in P - \bigcup_{n=1}^{\infty}[a_n, b_n]$ let $g(x) = h(x)$. It remains to define g on each $[a_n, b_n]$. Fix n. If there is no $m < n$ with $b_n < a_m$, then let $v = 1$. If there are such m, let k be such that $a_k = \min\{a_m : m < n\}$ and, in this case, let $v = h(b_k)$. Now define g on $[a_n, b_n]$ so that $g(a_n) = g(b_n) = h(b_n)$, $h(b_n) \leq g \leq v$ on $[a_n, b_n]$, g differentiable on (a_n, b_n), $g'_+(a) = 0$, and $g = v$ on the interval in the middle of $[a_n, b_n]$ of length $(1 - n^{-1})(b_n - a_n)$. It is easy to check that g is continuous except at each a_n. At these points g is right continuous and left approximately continuous. So g is bounded and approximately continuous. Since $g \geq h$, $g'_+(x) = +\infty$ for all x in P, x not an a_n. For each n, $g'_+(a_n) = 0$. Since g is differentiable on each (a_n, b_n), g'_+ exists on $[0, 1)$. Finally, g'_+ is not of Baire class one since g'_+ has no point of continuity in P relative to P.

Since the function g of this example is bounded and approximately continuous, there is a continuous function f such that $f' = g$. Continuing for any $n \geq 2$ there is a continuous function f such that $f^{(n-1)} = g$ and hence $f_+^{(n)} = g'_+$. So for each n, nth unilateral derivatives need not be of Baire class one.

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