ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF THE SECOND ORDER DIFFERENCE EQUATION

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ABSTRACT. The second order difference equation

\( \Delta^2 x_n + p_n f(x_n) = 0 \)

is considered. The results give a necessary and sufficient condition for some solution of \( (E) \) to have asymptotic behavior \( x_n \sim C = \text{const. as } n \to \infty \).

Introduction. The asymptotic behavior of the solutions of second order differential equations have been considered by R. A. Moore and Z. Nehari [4], W. F. Trench [9], and P. Waltman [10]. The next results for nth order nonhomogeneous differential equations was given by T. G. Hallam [1, 2]. Similar problems with regard to second order difference equations were investigated by J. W. Hooker and W. T. Patula [3] and J. Popenda [7].

In this paper the asymptotic behavior of solutions of the second order difference equation

\( \Delta^2 x_n + p_n f(x_n) = 0 \)

will be considered. A necessary and sufficient condition for some solution \( x \) of \( (E) \) to have the asymptotic behavior

\[ \lim_{n \to \infty} x_n = C, \]

where \( C \) is a constant such that \( f(C) \neq 0 \), will be proved.

Let \( N \) denote the set of positive integers and \( R \) the set of real numbers. Throughout this paper it will be assumed that \( f: R \to R \) is continuous and \( p: N \to R_+ \cup \{0\} \).

For a function \( a: N \to R \) we introduce the difference operator \( \Delta \) by

\[ \Delta a_n = a_{n+1} - a_n, \quad \Delta^2 a_n = \Delta(\Delta a_n), \]

where \( a_n = a(n) \), \( n \in N \). Moreover let \( \sum_{j=k}^{k-1} a_j = 0 \). One can observe that if \( f \) is definite and finite on \( R \) then \( (E) \) possesses solutions for any two initial values \( x_1, x_2 \in R \).

1. A necessary condition.

Theorem 1. A necessary condition for the existence of a solution \( x \) of \( (E) \) which possesses asymptotic behavior \( (AB) \) is

\[ \sum_{j=1}^{\infty} j p_j < \infty. \]
PROOF. Let $x$ denote a solution of (E) having the property (AB), i.e. $x_n \to C$ for $n \to \infty$. Then
\begin{equation}
\Delta x_n \to 0 \quad \text{as } n \to \infty.
\end{equation}
Assume that $f(C) > 0$. (The case $f(C) < 0$ with some modifications can be considered in a similar way.) The continuity of $f$ implies that there exists $\varepsilon > 0$ such that $f(t) > 0$, $t \in I := [C - \varepsilon, C + \varepsilon]$ for some $\varepsilon > 0$. Since $x_n \to C$ as $n \to \infty$, there exists $n_1 = N(\varepsilon)$ such that for each $n \geq n_1$, $x_n \in [C - \varepsilon, C + \varepsilon]$. Therefore
\[ f(x_n) \geq C_0 := \min_{t \in I} f(t) > 0 \quad \text{for } n \geq n_1. \]
Hence
\begin{equation}
\Delta x_n - \Delta x_k = - \sum_{j=k}^{n-1} p_j f(x_j) \leq -C_0 \sum_{j=k}^{n-1} p_j \quad \text{for } k \geq n_1.
\end{equation}
Using (1.1) we get
\begin{equation}
C_0 \sum_{j=k}^{\infty} p_j \leq \Delta x_k \quad \text{for } k \geq n_1.
\end{equation}
Therefore the series $\sum_{j=k}^{\infty} p_j$ is convergent. Summing (1.2) over $n$ and tending to infinity with an upper limit we yield
\begin{equation}
C_0 \sum_{j=n_1}^{\infty} \sum_{i=j}^{\infty} p_i \leq C - x_{n_1}.
\end{equation}
From this fact it follows that the series $\sum_{j=n_1}^{\infty} \sum_{i=j}^{\infty} p_i$ converges. Since
\[ \sum_{j=n_1}^{\infty} \sum_{i=j}^{\infty} p_i = \sum_{j=n_1}^{\infty} (j + 1 - n_1) p_j, \]
the series $\sum_{j=n_1}^{\infty} (j + 1 - n_1) p_j$ is also convergent. By observing that
\[ \sum_{j=n_1}^{\infty} j p_j = \sum_{j=n_1}^{\infty} (j + 1 - n_1) p_j + (n_1 - 1) \sum_{j=n_1}^{\infty} p_j, \]
we see that the series $\sum_{j=n_1}^{\infty} j p_j$ is convergent. Q.E.D.

REMARK 1. From (1.2) it follows that $\Delta x_k \geq 0$ for $k \geq n_1$. Therefore the solution $x_n$ is increasing for $n \geq n_1$. We see that $x_l \leq C$ for $l \geq n_1$. This result means that if $f(C) > 0$ then the solution of (E) which possesses the asymptotic behavior (AB) monotonically approaches $C$ from below. If $f(C) < 0$, then $x_n$ must monotonically tend to $C$ from above.

2. A sufficient condition.

THEOREM 2. For every $k \in N$ let
\begin{equation}
(*) \quad i_R + p_k f: R \to R \text{ be a surjection (} i_R \text{ denotes an identity function on } R). \end{equation}
A sufficient condition for the existence of a solution $x$ of (E) which possesses the asymptotic behavior (AB) is (NS).

PROOF. The cases $C > 0$ and $f(C) > 0$ will be considered. (The other cases, i.e. $C < 0$ or $f(C) < 0$, with some modifications can be shown in a similar way.)
Let (NS) hold. Hence

\[(2.1) \quad \lim_{n \to \infty} \sum_{j=n}^{\infty} jp_j = 0.\]

One can observe that the sequence \(\{\sum_{j=n}^{\infty} jp_j\}_{n=1}^{\infty}\) is nonincreasing. Analogous to the proof of Theorem 1 there exists an interval \(I = [C-\varepsilon, C+\varepsilon]\) such that \(f(t) > 0, t \in I\) for some \(\varepsilon > 0\). Denoting \(C_1 := \max_{t \in I^-} f(t)\), where \(I^- = [C-\varepsilon, C]\) from (2.1), we obtain

\[C_1 \sum_{j=n}^{\infty} jp_j \leq \varepsilon \quad \text{for all } n \geq N(\varepsilon).\]

Let us set

\[n_2 = \min \left\{ n \in N : C_1 \sum_{j=n}^{\infty} jp_j \leq \varepsilon \right\}.\]

Let \(l_\infty\) denote the Banach space of bounded sequences \(x = \{h_i\}_{i=1}^{\infty}\) with norm \(\|x\| = \sup_{i \geq 1} |h_i|\). Moreover let us define the set \(T \subset l_\infty\) in the following way:

\[x = \{h_i\}_{i=1}^{\infty} \in T \quad \text{if} \quad \begin{cases} h_k = C & \text{for } k = 1, 2, \ldots, n_2 - 1, \\ h_k \in I_k^- & \text{for } k \geq n_2, \end{cases}\]

where

\[I_k^- := \left[ C - C_1 \sum_{j=k}^{\infty} jp_j, C \right], \quad k \geq n_2.\]

It is easy to show that \(T\) is bounded, convex and closed in \(l_\infty\). We will show that \(T\) is compact. Set \(\text{diam}[a, b] = b - a; \ a, b \in R\). By (NS) it follows that \(\text{diam} I_k^- \to 0\) for \(n \to \infty\). Choose any \(\varepsilon_1 > 0\). If \(\varepsilon_1\) is such that \(\text{diam} I_{n_2}^- < \varepsilon_1\), then the element \(v = \{C, C, C, \ldots\} \in l_\infty\) is an \(\varepsilon_1\)-net. The case \(\text{diam} I_{n_2}^- \geq \varepsilon_1\) will be considered.

Let \(n_3 \geq n_2\) be such that \(\text{diam} I_{n_3}^- \geq \varepsilon_1\) and \(\text{diam} I_{n_3+1}^- < \varepsilon_1\). (Everyone can find \(n_3\) because \(\text{diam} I_n^- \to 0\) for \(n \to \infty\).) Then it is easy to show that the set of elements of the space \(l_\infty\) in the form

\[v_{s_1, s_2, \ldots, s_{n_3-n_2+1}}^{1, 2, \ldots, n_3-n_2+1} = \{C, \ldots, C, \ldots, C - s_1 C_1, \ldots, C - s_{n_3-n_2+1} C_1, C, \ldots\}\]

where

\[s_i = 0, 1, \ldots, r_i := \text{En} \left[ \frac{\text{diam} I_{n_2+i-1}^-}{\varepsilon_1} \right] + 1, \quad i = 1, 2, \ldots, n_3 - n_2 + 1,\]

to set up an \(\varepsilon_1\)-net. (En denotes an entire function.) One can observe that

\[\text{card}\{v_{s_1, s_2, \ldots, s_{n_3-n_2+1}}^{1, 2, \ldots, n_3-n_2+1}\} = \prod_{i=1}^{n_3-n_2+1} (r_i + 1) < \infty.\]

Hence the \(\varepsilon_1\)-net is finite and by the Hausdorff theorem \(T\) is compact.

Let us define the operator \(A\) on \(T\) in the following way:

\[Ax = y = \{b_1, b_2, \ldots, b_{n_2-1}, b_{n_2}, \ldots, b_k, \ldots\},\]
where

\[
b_n = \begin{cases} 
C & \text{for } n = 1, 2, \ldots, n_2 = 1; \\
C - \sum_{j=n}^{\infty} (j + 1 - n)p_j f(h_j) & \text{for } n \geq n_2.
\end{cases}
\]

We will show that \( A \) is a function from \( T \) to \( T \). By observing that \( I_k^- \subset I^- \) it follows that \( 0 < f(h_k) \leq C_1 \) for \( k \geq n_2 \). For \( j \geq k \) one obtains the inequality

\[
0 < (j + 1 - k)p_j f(h_j) \leq jp_j f(h_j) \leq C_1 j p_j.
\]

Hence

\[
C > C - \sum_{j=k}^{\infty} (j + 1 - k)p_j f(h_j) \geq C - C_1 \sum_{j=k}^{\infty} j p_j.
\]

Thus \( b_k \in I_k^- \) for \( k \geq n_2 \). Therefore \( y \in T \).

Next we will show that \( A \) is continuous. Since \( f \) is continuous on \( R \), it is uniformly continuous on \( I^- \). Hence for each \( \varepsilon_2 > 0 \) there exists \( \delta_1 > 0 \) such that the condition \( |t_1 - t_2| < \delta_1 \) implies \( |f(t_1) - f(t_2)| < \varepsilon_2 \). Consider the sequence \( \{x_m\}_{m=1}^{\infty} \), \( x_m \in T \), such that

\[
(2.2) \quad \|x_m - x^0\| \to 0; \quad \text{i.e., sup}_{n \geq 1} |h_n^m - h_n^0| \to 0, \quad \text{as } m \to \infty.
\]

From (2.2) it follows that there exists \( n_3 = N(\delta_1) \) such that

\[
\|x_m - x^0\| < \delta_1; \quad \text{i.e., sup}_{n \geq 1} |h_n^m - h_n^0| < \delta_1 \quad \text{for } m \geq n_3.
\]

Hence

\[
\forall_{m \geq n_3} \forall_{i \in N} |h_i^m - h_i^0| < \delta_1.
\]

Then for \( m \geq n_3 \)

\[
\|Ax_m - Ax^0\| = \sup_{n \geq 1} |b_n^m - b_n^0|
\]

\[
= \sup_{n \geq n_2} \left| \sum_{j=n}^{\infty} (j + 1 - n)p_j f(h_j^m) - \sum_{j=n}^{\infty} (j + 1 - n)p_j f(h_j^0) \right|,
\]

where \( b^0 = Ax^0 \) and \( b^m = Ax^m \).

Since the series \( \sum_{j=n}^{\infty} (j + 1 - n)p_j f(h_j^m) \), \( \sum_{j=n}^{\infty} (j + 1 - n)p_j f(h_j^0) \) are convergent,

\[
\|Ax_m - Ax^0\| \leq \varepsilon_2 \sum_{j=n_2}^{\infty} (j + 1 - n_2)p_j, \quad m \geq n_3.
\]

Hence \( A \) is continuous.

By the Schauder fixed point theorem [8] there exists a solution in \( T \) of the equation \( x = Ax \). Let \( z = \{d_1, d_2, \ldots, d_{n_2-1}, d_{n_2}, \ldots\} \) denote such a solution. Since \( z \in T \), it can be written as follows:

\[
z = \{C, C, \ldots, C, d_{n_2}, d_{n_2+1}, \ldots\}
\]
and

\[ Az = \left\{ C, C, \ldots, C, C - \sum_{j=n_2}^{\infty} (j + 1 - n_2)p_j f(d_j), \right. \]
\[ \left. C - \sum_{j=n_2+1}^{\infty} (j - n_2)p_j f(d_j), \ldots \right\}. \]

Therefore

\[ (2.3) \quad d_n = C - \sum_{j=n}^{\infty} (j + 1 - n)p_j f(d_j) \quad \text{for } n \geq n_2. \]

Applying the operator \( \Delta \) to (2.3) we yield

\[ \Delta d_n = \sum_{j=n}^{\infty} p_j f(d_j) \quad \text{for } n \geq n_2. \]

Hence \( \Delta^2 d_n = -p_n f(d_n) \) holds for \( n \geq n_2 \). This means that the sequence \( \{d_n\}_{n=1}^{\infty} \) fulfills the equation (E) but for \( n \geq n_2 \) only.

We now prove the existence of the solution \( \{x_n\}_{n=1}^{\infty} \) of (E) such that \( x_n = d_n \) for \( n \geq n_2 \).

One can observe that (E) can be rewritten as

\[ x_n + p_n f(x_n) = -x_{n+1} + 2x_{n+1}. \]

If \( n = n_2 - 1 \) we get

\[ (2.4) \quad x_{n_2-1} + p_{n_2-1} f(x_{n_2-1}) = -x_{n_2+1} + 2x_{n_2}. \]

But we demand for \( x_n \) to be equal to \( d_n \) for \( n \geq n_2 \).

From (2.4) we obtain

\[ x_{n_2-1} + p_{n_2-1} f(x_{n_2-1}) = -d_{n_2+1} + 2d_{n_2}. \]

By (*) it follows that the equation

\[ x + p_{n_2-1} f(x) = -d_{n_2+1} + 2d_{n_2} \]

possesses solutions. Let us denote one of them by \( x_{n_2-1} \). Analogously we can calculate \( x_{n_2-2}, x_{n_2-3}, \ldots, x_2, x_1 \) one after the other. Consequently we get the sequence which fulfills (2.4), i.e. which also fulfills (E). Moreover this sequence is identical to \( \{d_n\}_{n=1}^{\infty} \) for \( n \geq n_2 \) and it has the asymptotic behavior (AB) because \( \lim_{n \to \infty} d_n = C \). Q.E.D.

**REMARK 2.** One can observe that if \( f \) is bounded on \( R \) or fulfills the condition \( x f(x) > 0 \) for \( x \neq 0 \) then condition (*) is satisfied. From the proof of Theorem 2 we can deduce that (*) may be weakened as follows:

\[ i_R + p_k f : R \to R \quad \text{for } k < n_2, \quad k \in N. \]

**REMARK 3.** If the assumptions of Theorem 2 hold then analogously an existence of a solution of the equation

\[ (E_k) \quad \Delta^2 x_n + p_{n+k} f(x_{n+k}) = 0, \quad k \geq 1, \]
having the asymptotic behavior (AB) may be proved. In this case the operator $A$

similar to the above but with

$$b_n = C - \sum_{j=n+k}^{\infty} (j + 1 - n - k)p_j f(h_j) \text{ for } x = \{h_i\}_{i=1}^{\infty} \in T$$

should be defined.

REMARK 4. If (E) possesses a solution $x$ such that $\lim_{n \to \infty} x_n = C$ then equation

(E) has a solution with $\lim_{n \to \infty} x_n = C_2$, where $C_2 \in (C - \varepsilon, C + \varepsilon) \subset I$.

REMARK 5. If for some $C$, $f(C) = 0$, then independently of the form of $p$,

equation (E) has a solution with (AB). It has the form $x_n = C$ for each $n \geq 1$.

Conversely, if, for each $n \geq n_2$, $x_n = C$ is the solution of (E) then $p_n f(C) = 0$ for

$n \geq n_2$. Hence $f(C) = 0$ or $p_n = 0$ for each $n \geq n_2$. For the second case ($p_n = 0$)
the condition $\sum_{j=1}^{\infty} j p_j < \infty$ obviously holds.

EXAMPLE. The special case $f(x) = x$ and $k = 1$ will be studied. In this case

equation (E_k) can be written in the following two equivalent forms:

$$(E_1) \quad \Delta^2 x_n + p_{n+1} x_{n+1} = 0, \quad x_{n+2} - q_n x_{n+1} + x_n = 0,$$

where $q_n = 2 - p_{n+1}$, $n \in N$. If $q_n < 2$, $n \in N$ and $\sum_{j=2}^{\infty} (2 - q_{j-1}) j < \infty$ then

(E_1) possesses a solution which asymptotically approaches any positive constant.

Analogously in the case $k = 2$ one obtains the equation

$$(E_2) \quad x_{n+2} - 2 q_n x_{n+1} + q_n x_n = 0,$$

where $q_n = 1/(p_{n+2} + 1)$.

If $0 < q_n < 1$ and $\sum_{j=3}^{\infty} (1/q_{j-1} - 1) j < \infty$ then (E_2) possesses a solution which
asymptotically approaches any positive constant.

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