EXPOSITIONATION OF REALS: EFFECTS OF BASE CHOICE

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ABSTRACT. Let \( r > 1 \) and \( X_r \) be the minimal set of reals containing 1 and closed under \( \exp_r : x \mapsto r^x \) and addition. The behavior of \( X_r \) is studied. In particular among possible order types of \( X_r \) there are \( \omega, \omega^\omega, \omega + q, \omega + 1 + q, \) where \( q \) is the dense countable order without endpoints.

Let \( r > 0 \) and \( X_r \) be the minimal set of reals containing 1 and closed under \( \exp_r : x \mapsto r^x \) and addition. Obviously \( X_r \) is dense in \( \mathbb{R}^+ \) for \( r < 1 \) and \( X_1 = \mathbb{N} \). I would like to describe the behavior of \( X_r \) for \( r > 1 \).

Preliminary remarks. (Proofs are provided in §1.) (i) \( X_r = \) the set of values of constant \((1, +, -, \exp_r)\)-terms.

(ii) If \( X \) is a well-ordered subset of \( \mathbb{R}^+ \) closed under addition (or multiplication) then the order type of \( X \) is \( \omega^\alpha \) for some \( \alpha \).

(iii) For any countable \( \alpha \) if \( X_r \) is well-ordered and its order type is \( < \omega^\alpha \) then there exists a set \( X \subset \mathbb{R}^+ \) containing \( X_r \), closed under +, -, \( \exp_r \) and of order type \( \omega^\alpha \).

PROPOSITION 1. Let \( a = (a^a + 1)^{1/(a^a + 1)} \) and \( h = s^{1/s} \) where \( s \) is the root of the equation \( x^{1/x} = (x + 1)^{1/(x+1)} \). (One can easily see that both equations have unique roots.) Then

(i) If \( e^{1/e} < r \) then \( X_r \) has order type \( \omega \).

(ii) If \( a \leq r \leq e^{1/e} \) then \( X_r \) and \( X_\alpha \) have order type \( \omega^\alpha \).

(iii) If \( b < r < a \) then \( X_r \) is not well-ordered and the Cantor derivative \( (X_r)^{(\omega)} = \emptyset \).

(iv) \( X_b \) is not well-ordered.

The Cantor derivative mentioned in (iii) is defined as follows: \( Y(0) = \overline{Y}, Y(\lambda + 1) = \{Y(\mu) | \mu < \lambda \} \) for limit \( \lambda \).

Some numerical values:

\[ e^{1/e} = 1.4446678..., \quad a = 1.4446575..., \quad b = 1.4360782.... \]

CONJECTURE. If \( 1 < r < b \) then \( X_r = \{a \text{ sequence increasing to } g_r \} \cup [g_r, \infty) \), where \( g_r \) is the smallest root of the equation \( r^x = x \).

PROPOSITION 2. (i) If \( 1 < r < b \) then \( X_r \) contains a perfect subset i.e. a nonempty subset without isolated points.

(ii) The conjecture is valid for \( 1 < r < 1.4360782... \).

(iii) If \( \ln(\ln(g_r))/\ln(\ln(g_r + 1)) \) is irrational then the conjecture is valid for \( r = b \).

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Ends of proofs of claims and lemmata are marked with \( \square \).

1. Proofs of preliminary remarks. (i) I have to construct for any constant 
\((1,+,\cdot,\exp_r)\)-term \(t\) a constant 
\((1,+,\exp_r)\)-term \(u\) with the same value. Proceed 
by induction on the number of operations in \(t\). Let \(t = t_1 t_2\), where \(t_1, t_2\) are 
\((1,+,\exp_r)\)-terms and \(t_1 \neq t_2\). If e.g. \(t_2 = u + v\) then we can apply the 
induction assumption to \(t_1 u, t_1 v\) in \((t_1 u) + (t_1 v)\). If both \(t_1, t_2\) are exponents, 
\(t_1 = r^u, t_2 = r^v\), then the value of \(t\) is the value of \(\exp_r(u + v)\). \( \square \)

(ii) Let \(\tau\) be the order type of \(X\). It suffices to prove that if \(\omega^\lambda < \tau\) then 
\(\omega^{\lambda 2} < \tau\) for any \(\lambda\), see \([2, \text{VII}.7]\). Let \(X\) be closed under addition (the case of \(X\) closed under 
multiplication is quite similar). Let \(A\) be a bounded subset of \(X\) with order type 
\(\omega^\lambda\) and \(x = \sup(A)\); for each \(a \in A\) let \(a^+\) be its successor. Then for any \(a \in A\) the 
order type of \(\{a^+ + b | b \in A & a + x < a^+ + b\}\) is \(\omega^\lambda\), see \([2, \text{VII}.6]\) (any nonzero 
remainder of \(\omega^\lambda\) is \(\omega^\lambda\)). Thus \(A + A\) has order type at least \(\omega^{\lambda 2}\) and is bounded by 
\(2x\). \( \square \)

(iii) I will prove the following statement: for any set \(A \subset \mathbb{R}_+\) containing 1 and 
closed under \(+,\cdot,\exp_r\) with order type \(\omega^{\alpha}\), for any \(p \in \mathbb{R}_+\), for any countable 
ordinal \(\beta\) there exists a set \(B \subset \mathbb{R}_+\) closed under \(+,\cdot,\exp_r\) with order type \(\omega^{\alpha + \beta}\) 
and satisfying \(B \supset A \& A \supset B \cap (0,p)\).

Proceed by induction on \(\beta\). Consider first the limit case. Let \(p > 0\) and \(\beta = \lim\{\beta_i + \cdots + \beta_n\}_n\) with nonzero \(\beta_i\)'s. Pick up \(\{B_i\}_i\) as follows: let \(B_0 = A\) and 
\(B_{i+1}\) be a set \(\subset \mathbb{R}_+\) such that \(B_{i+1} \supset B_i\), \(B_{i+1} \cap (0,p + i + 1) \subset B_i\), \(B_{i+1}\) is 
closed under \(+,\cdot,\exp_r\) and its order type is \(\omega^{\lambda_i}\), where \(\lambda_i = \omega^{\alpha + \beta_1 + \cdots + \beta_{i+1}}\). Put 
\(B = \bigcup B_i\). Since any bounded subset of \(B\) is contained in one of \(B_i\)'s \(B\) is well-
ordered and has order type at most \(\omega^{\alpha + \beta}\), and since \(B\) contains all \(B_i\)'s its order 
type is at least \(\omega^{\alpha + \beta}\).

Now turn to the case of \(\beta = 1\). Without loss of generality we can assume that 
\(p > 1\) and \(r^x > x + 1\) for \(x \geq p\). Let \(K\) be a subset of \((p,p + 1)\) with order type 
\(\omega^{\alpha}\) and \(B\) be the minimal subset of \(\mathbb{R}_+\) closed under \(+,\cdot,\exp_r\) and containing 
\(A \cup K\). Since any bounded subset of \(B\) is contained in a finite union of \((+,\cdot,\exp_r)\)- 
combinations of \(A,K\) the result follows from \([3]\):

Let \(X,Y \subset \mathbb{R}_+\) be well-ordered; then the order type of \(X \cup Y\) 
is \(\leq\) the Hessenberg natural sum of the order types of \(X\) and 
\(Y\), and the order types of \(X + Y\) and \(X \cdot Y\) are \(\leq\) the Hessenberg 
natural product of the order types of \(X\) and \(Y\).

(This result is not very difficult, and you can prove it for your own pleasure.) \( \square \)

2. Proof of Proposition 1. For \(r < e^{1/e}\) let \(g_r < h_r\) be the roots of the 
equation \(x^{1/x} = r\) (for \(r = e^{1/e}\) there is unique root \(x = e\) and for \(r > e^{1/e}\) there are 
no roots). Clearly \(g_r < e < h_r\), both \(g_r\) and \(h_r\) depend on \(r \in (1,e^{1/e})\) differentiably 
and monotonically and tend to \(e\) for \(r \to e^{1/e}\), and \(r^x > x\) for \(x \notin [g_r,h_r]\) and \(r^x < x\),
for \( x \in (g_r, h_r) \). For convenience put also \( g_r = h_r = e \) for \( r = e^{1/e} \). Denote \( \exp_r^0 : x \mapsto x \) and \( \exp_r^{n+1}(x) = \exp_r(\exp_r^n(x)) \). The sequence \( \{\exp_r^n(1)\}_n \) increases and if bounded tends to \( g_r \).

Claim 1. Let \( T \) be an infinite set of constant \((1, g_r, +, \exp_r)\)-terms. If the set of values of \( t \in T \) is bounded then \( T \) contains terms with arbitrarily high \( \exp_r \)-nests.

Proof. If values of \( t \in T \) are bounded then there exists an \( n \) such that each \( t \in T \) consists of at most \( n \) summands of the forms \( 1, g_r, r^u \). Thus the set of such summands is still infinite and the values of \( u \)’s are bounded.

This provides (i) of the proposition: if \( t \) contains an \( \exp_r \)-nest of height \( n \) then \( t \geq \exp_r^n(1) \), and \( \{\exp_r^n(1)\}_m \) is unbounded for \( r > e^{1/e} \). On the other hand if \( r \leq e^{1/e} \) then \( g_r \) is a limit point for \( X_r \) and since \( X_r \) is closed under addition all its finite number Cantor derivatives are nonempty.

Claim 2. Let \( b < r \leq e^{1/e} \) and \( \{t_n\}_n \) be a sequence of constant \((1, g_r, +, \exp_r)\)-terms whose values are bounded. Then there exist finite sets \( U \) of constant \((1, +, \exp_r)\)-terms and \( V \) of terms in variables and \( 1, g_r, +, \exp_r \) such that each \( t_n = v(w_1, \ldots, w_l) \) for some \( v \in V \), where each \( w_i = \exp_r^n(u) \) for some \( u \in U \cup \{g_r\} \) and \( m \).

Proof. Note first that there is a \( k \) such that \( \exp_r^k(1) + 1 > h_r \) (this follows from \( r > b \): \( h_b = 1 + g_b = 1 + \lim \{\exp_r^n(1)\}_m \)). Let \( K \) be the least such \( k \). Note also that \( \{\exp_r^n(x)\}_m \) is unbounded for \( x > h_r \). Let the values of \( t_n \)’s be \(< M \) and \( J \) be the least \( j \) such that \( \exp_r^j(\exp_r^k(1) + 1) > M \).

Let \( U = \{1\} \cup \{u | u \text{ is a sum and appears as a subterm of some } t_n \text{ in the scope of at least } J \exp_r \text{’s} \} \). I will show that \( U \) is finite. Note that the values of \( t \in U \) are bounded by \( \exp_r^K(1) + 1 \). If \( U \) was infinite then Claim 1 would tell us that elements of \( U \) contain arbitrarily high \( \exp_r \)-nests which is impossible in view of the above bound and the fact that the nontrivial elements of \( U \) are sums. If a term \( u \in U \) contained \( g_r \), we would have \( u = q + s \geq g_r + 1 > \exp_r^K(1) + 1 \); thus elements of \( U \) are \((1, +, \exp_r)\)-terms.

To obtain \( V \) imagine \( t_n \)’s typewritten on sheets of paper so that each \( \exp_r(w) \) is displayed as \( r^w \) and substitute by variables the parts of \( t_n \)’s which are above \( J \).

One can express this rigorously as follows. For a subterm \( u \) of \( w \) let \( \text{height}(u, w) \) = the number of \( \exp_r \)’s in \( w \) affecting \( u \); and let \( \text{height}(w) = 1 + \max \{\text{height}(u, w) | u \text{ is a subterm of } w \} \). Present each \( t_n \) with height \( J \) in the form \( v(w_1, \ldots, w_l) \), where \( v \) is a term in variables and \( 1, g_r, +, \exp_r \); \( w_i \)’s are constant \((1, g_r, +, \exp_r)\)-terms, and \( \text{height}(v) = J + 1 \), \( \text{height}(\text{variable}, v) = \text{height}(w_i, t_n) = J \) for each \( i \).

Let \( V \) be the set of such \( v \)’s. To show that \( V \) is finite note first that the values of \( \{v(1, \ldots, 1) | v \in V \} \) are bounded \((< M) \); now reasoning similar to that in the proof of Claim 1 provides the result.

Corollary 3. If \( b < r \leq e^{1/e} \) then \( (X_r)^{(m)} = \{\text{the values of constant } (1, g_r, +, \exp_r) \text{-terms with } \geq m \text{ occurrences of } g_r \} \) for any \( m \), and \( (X_r)^{(\omega)} = \emptyset \).

Proof. Proceed by induction on \( n \). Let \( x \in (X_r)^{(n+1)} \),

\[ x = \lim \{\text{value of } t_m\}_m, \]

each \( t_m \) being a constant \((1, g_r, +, \exp_r)\)-term with \( \geq n \) occurrences of \( g_r \) and
the values of \( t_m \)'s distinct. In view of Claim 2 we can assume that each \( t_m = u(v_m, \ldots, w_m) \) for a term \( u \) in variables and \( 1, g_r, +, \exp_r \) with \( n \) occurrences of \( g_r \) and \( v_m = \exp_r^{j(m)}(v), \ldots, w_m = \exp_r^{k(m)}(w) \) for some constant \((1, +, \exp_r)-\)terms \( v, \ldots, w \) with increasing \( j(), \ldots, k() \). Thus \( x \) is the value of \( u(g_r, \ldots, g_r) \). Conversely the value of \( t(g_r) \) is \( \lim \{\text{value of } t(\exp_r^m(1))\}_m \). Besides, Claim 2 shows that if \( \{t_m\}_m \) is a bounded sequence of constant \((1, g_r, +, \exp_r)-\)terms then \{the number of occurrences of \( g_r \) in \( t_m \)\}_m is bounded, whence \( (X_r)^{(\omega)} = \emptyset \). \( \square \)

COROLLARY 4. If \( b < r \leq e^{1/e} \) and \( X_r \) is well-ordered then its and \( \overline{X}_r \)'s order types are both \( \omega^\omega \).

PROOF. Immediately from the preliminary remark (ii) and the above corollary. \( \square \)

Clearly \( r^{r+1} < r^r + 1 \) for \( r < a \) whence \( \{\exp_r^n(r^r + 1)\}_m \) decreases for \( r < a \) which gives (iii), (iv) of the proposition. I will show that \( X_r \) is well-ordered for \( r \geq a \).

Claim 5. If \( X_r \) is not well-ordered then there exists a sequence \( \{t_m\}_m \) of \((1, +, \exp_r)-\)terms such that \( t_m > \exp_r(t_{m+1}) \) for each \( m \).

PROOF. Let \( \{s_m\}_m \) be a decreasing sequence of \((1, +, \exp_r)-\)terms. Put \( t_0 = s_0 \). Since the values of \( \{s_m\}_m \) are bounded \{the number of summands in \( s_m \)\}_m is also bounded. Consequently (turning if necessary to a subsequence) we may assume that there exists a sequence \( \{s'_m\}_{m \geq 1} \) of \((1, +, \exp_r)-\)terms such that each \( \exp_r(s'_m) \) is a summand of \( s_m \) and the values of \( \{s'_m\}_{m \geq 1} \) decrease. Of course the value of \( t_0 \) is greater than those of \( \{\exp_r(s'_m)\}_{m \geq 1} \). Apply the same procedure to \( \{s'_m\}_{m \geq 1} \) and proceed \( \omega \) times. \( \square \)

Claim 6. Let \( b < r \leq e^{1/e} \). If \( X_r \) is not well-ordered then there exists a \((1, +, \exp_r)-\)term \( t \) such that \( r^t < t \).

PROOF. Let \( \{t_m\}_m \) be a sequence of \((1, +, \exp_r)-\)terms such that \( t_m > \exp_r(t_{m+1}) \) for each \( m \) (see Claim 5). If \( t_m > h_r \) then \( \&_k t_{m+k} < x_k \), where \( x_{k+1} = \log_r(x_k) \), \( x_0 = \) the value of \( t_m \); and \( \{x_k\}_k \) converges to \( h_r \). Thus the values of \( t_m \)'s are bounded. Since for \( k > 0 \) inequality \( \exp_r^k(x) < x \) holds if \( x \in (g_r, h_r) \) it will be enough to find \( t \) with \( t > \exp_r^k(t) \) for some \( k > 0 \): then \( g_r < t < h_r \) and \( r^t < t \).

If for some \( m < n \) we have \( t_m = t_n \) then we have \( t_m > \exp_r^{n-m}(t_m) \). Suppose that all \( t_m \)'s are distinct. Then by Claim 2 there exist an increasing sequence \( s \) of positive integers, a term \( v \) in variables and \( 1, +, \exp_r, \) constant \((1, +, \exp_r)-\)terms \( u_1, \ldots, u_l \) and increasing sequences \( k_1, \ldots, k_l \) of positive integers such that for each \( m \) we have \( t_s(m) = v(\exp_r^{k_1(m)}(u_1), \ldots, \exp_r^{k_l(m)}(u_l)) \). If \( \exp_r^{k_1(1)}(u_1) > \exp_r^{k_2(1)}(u_1) \) for at least one \( i \) then we have \( u_i > \exp_r^{k_i(1) - k_l(1)}(u_i) \). Otherwise we have

\[
t_{s(1)} > \exp_r^{k_1(1) - s(1)}(t_{s(2)})
= \exp_r^{k_1(2) - s(1)}(v(\exp_r^{k_2(2)}(u_1), \ldots, \exp_r^{k_l(2)}(u_l)))
\geq \exp_r^{k_1(1) - s(1)}(v(\exp_r^{k_1(1)}(u_1), \ldots, \exp_r^{k_l(1)}(u_l)))
= \exp_r^{k_1(1) - s(1)}(t_{s(1)}).
\]

Claim 7. If \( a \leq r < e^{1/e} \) then \( X_r \) is well-ordered.

PROOF. Suppose the contrary; let \( t \) be a \((1, +, \exp_r)-\)term such that \( r^t < t \). Without loss of generality we can assume that \( t \) is not an exponent (if \( t = r^u \) then
$r^u < r^u$ implies $r^u < u)$. Consequently $t$ is a sum. If $t$ has three or more summands then its value is $\geq 3 > h_a \geq h_r$ contradicting the choice of $t$ (for $x > h_r$ implies $r^x > x$). Thus $t$ has two summands. If both summands of $t$ are $\neq 1$ then they are $\geq r$ whence $t \geq 2r \geq 2.889 \ldots > h_a \geq h_r$ (again impossible). On the other hand $2 < g_a < g_r$ whereas $r^x > x$ for $x < g_r$. Thus $t = 1 + r^u$. If $u$ is a sum then $t \geq 1 + r^2 > 1 + 2 > h_a \geq h_r$. If $u = 1$ then the value of $t$ is $1 + r \leq 2.444 \ldots < g_a \leq g_r$. Thus $u = r^w$. If $v$ is a sum then $t \geq 1 + r^v > 1 + r^2 > 3 > h_a \geq h_r$. If $v = r^w$ then $t \geq 1 + r^v > 1 + a^w = 2.8699218 \ldots > h_a \geq h_r$. Thus $u = 1$ and $t = 1 + r^r$. However $\exp_r(1 + r^r) \geq 1 + r^r$ for $r \geq a$ by definition of $a$. □


**LEMMA 1.** Let $f \in C^1([p, q])$, $f(p) = p$, $f(q) \geq p + 1$ and $0 < f'(x) < 1$ for $x \in (p, q)$. Let $Y$ be the subset of $[p, q]$ minimal with respect to the properties: $p \in Y$, $x \in Y$, $x + 1 < q \Rightarrow x + 1 \in Y$, $x \in Y \Rightarrow f(x) \in Y$. Then $Y = [p, q]$.

**Proof.** Suppose the contrary. Any open interval $C(p, q) - Y$ will be called a hole. Since $Y$ is closed under $+1$, $f$ we have $Y \supset [p, q] \cap (\bigcup_{m=0}^{\infty} m + f(Y))$ whence the length of any hole is bounded by that of a hole $C(p, p + 1) \subset (p, f(q))$. On the other hand $f$ maps $(p, q)$ onto $(p, f(q))$ monotonically with derivative $< 1$, so any hole $C(p, f(q))$ is the $f$-image of a larger one. Thus we obtain a contradiction: let $H$ be a hole with maximal length, let $H'$ be a hole $C(p, p + 1)$ with length($H') \geq$ length($H$), let $H'' = f^{-1}(H')$, then length($H''$) $>$ length($H$). □

**COROLLARY 2.** Let $c = s^{-1/s}$, where $s$ is the root of the equation $x + 1 = x/\ln(x)$. Then $[g_r, \infty) \subset X_r$ for $1 < r < c$ ($c = 1.3998168 \ldots$).

**Proof.** Put $d = -\ln(\ln(r))/\ln(r)$. Then $(r^x)' = r^x \cdot \ln(r) < 1$ for $x < d$ and $g_r + 1 \leq r^d < d < h_r$ for $r < c$. So the result follows from Lemma 1 applied to $\exp_r \in C^1([g_r, d])$. □

Now let us turn to $r \in (c, b)$. A real $x \in [g_r + 1, \log_r(g_r + 1))$ will be called tame if it is the value of a constant $(1, g_r, +, \exp_r)$-term. Clearly $\{\text{tame reals}\} \subset X_r$.

**Claim 3.** If the set of tame reals is dense in $(g_r + 1, \log_r(g_r + 1))$ then $X_r$ is dense in $(g_r, \infty)$.

**Proof.** If tame reals are dense in $(g_r + 1, \log_r(g_r + 1))$ then $X_r$ is also dense there and consequently $X_r$ is dense in $\bigcup_{k=0}^{\infty} \exp_r^k((g_r + 1, \log_r(g_r + 1)))$ which is just $(g_r, \log_r(g_r + 1))$ since $g_r = \lim \{\exp_r^k(g_r + 1)\}_k$ for $r < b$. Thus $X_r$ is dense in $[g_r, g_r + 1) \subset (g_r, \log_r(g_r + 1))$ and so $X_r$ is dense in $(g_r, \infty)$. □

So, I intend to prove that tame reals are dense in $(g_r + 1, \log_r(g_r + 1))$ for $c \leq r < 1.4360782 \ldots$ and I conjecture that this will hold for any $r \in (c, b)$. For any sequence $m = \{m_k\}_{k \geq 1}$ of positive integers let $\{mL_k\}_k$ be the following sequence of one-variable $(1, +, \exp_r)$-terms: $mL_0(x) = x$, $mL_k(x) = mL_{k-1}(\exp_r^{m_k}(x) + 1)$ for $k \geq 1$.

**Claim 4.** For any $x \in (g_r + 1, \log_r(g_r + 1))$ there exist unique $l \in \mathbb{N}$ and $y \in [g_r + 1, \log_r(g_r + 1)]$ such that $x = \exp_l^1(y) + 1$. For any nontame $x \in [g_r + 1, \log_r(g_r + 1)]$ there exist unique sequences $\{m_k\}_{k \geq 1}$ of positive integers and $\{y_k\}_{k \geq 0}$ of reals in $(g_r + 1, \log_r(g_r + 1))$ such that $x = mL_k(y_k)$ for each $k$.

**Proof.** The first part of the claim is an immediate consequence of the facts that $\log_r(g_r + 1) - (g_r + 1) < 1$ for $r \geq c$ (existence) and that $(g_r, \log_r(g_r + 1))$ is the disjoint union of $\{\exp_r^l(g_r + 1, \log_r(g_r + 1))\}_{l=0}^{\infty}$ for $r < b$ (uniqueness). For the
second part of the claim proceed by induction: if we have already $y_k$ (put $y_0 = x$) then $y_k \neq g_r + 1$ (since $x$ is not tame) and by the first part of the claim we get $m_{k+1}$ and $y_{k+1}$. □

Claim 5. Tame reals are dense in $(g_r + 1, \log_r(g_r + 1))$ for $c \leq r < 1.4360782 \ldots$

(proof of Proposition 2).

Proof. Let $x \in (g_r + 1, \log_r(g_r + 1))$ be nontame and $\{m_k\}_k, \{y_k\}_k$ be as in Claim 4. We have $x = mL_k(y_k)$ for each $k$ and I am going to prove that $x = \lim \{mL_k(g_r + 1)\}_k$ Note that functions $\{mL_k\}_k$ and their derivatives are positive and increasing (except the trivial case $k = 0$). Since

$$mL_k(y_k) - mL_k(g_r + 1) < (y_k - g_r - 1) \cdot \left(\frac{d}{dt} mL_k(t) \bigg|_{t = y_k}\right)$$

and $\log_r(g_r + 1) - g_r - 1 < 1$ for $r \geq c$, it suffices to show that $(mL_k(t))' \big|_{t = y_k}$ tends to 0 while $k \to \infty$. Since

$$(mL_k(t))' \big|_{t = y_k} = (mL_{k-1}(\exp^{m_k(t) + 1})') \big|_{t = y_k} = (mL_{k-1}(t))' \big|_{t = y_k - 1} \cdot (\exp^{m_k(t) + 1})' \big|_{t = y_k}$$

and

$$(\exp^{m_k(t) + 1})' \big|_{t = y_k} < (\exp^{m_k(t) + 1})' \big|_{t = \log_r(g_r + 1)}$$

it will be enough to prove that $\sup \{((\exp^{m_k(t)})')' \big|_{t = \log_r(g_r + 1)}\}_k < 1$. Well,

$$((\exp^{m(t)})')' = \prod_{j=1}^n (\ln(r) \cdot \exp^{t_j}(t))$$

and its factors $\{\ln(r) \cdot \exp^{t_j}(t)\}_j$ decrease with increasing $j$ for any fixed $t \in (g_r, h_r)$. Consequently if $(\exp^{m(t)}) < 1$ at some $t \in (g_r, h_r)$ then $(\exp^{m(t)}) < 1$ at the same $t$ whenever $m > n$. Let $J = J(r)$ be the minimal positive integer with the property $\exp^{J}(g_r + 1) + 1 < \log_r(g_r + 1)$. Then $m_k \geq J$ for all $k$ and the claim will go through whenever $(\exp^{m_k(t)})' \big|_{t = \log_r(g_r + 1)} < 1$. I checked this by machine computations (with long reals) for $c \leq r < 1.4360782 \ldots$ (corresponding to $J \leq 150$). □

Remark. Actually I conjecture that the inequality $(\exp^{J(r)}(t))' \big|_{t = \log_r(g_r + 1)} < 1$ (see above) holds for all $r \in (c, b)$. Moreover, it is possible to extend $\{\exp^{J_k}\}_k$ to a one-parameter iteration subgroup of the diffeomorphisms group of $(g_r, h_r)$, and I conjecture that

$$\frac{d}{dr} \left(\frac{d}{dt} \exp^{J(r)}(t) \bigg|_{t = \log_r(g_r + 1)}\right) < 0$$

for $c \leq r < b$,

where $J(r)$ is a differentiable function of $r$ determined by equation $\exp^{J(r)}(g_r + 1) + 1 = \log_r(g_r + 1)$.

Claim 6. There exists $K \in \mathbb{N}$ such that $(\exp^{M}(t))' \big|_{t = \log_r(g_r + 1)} < 1$ for $M \geq K$.

Proof. $(\exp^{M}(t))' = \prod_{j=1}^M (\ln(r) \cdot \exp^{t_j}(t))$ and $\lim \{\ln(r) \cdot \exp^{J-1}(g_r + 1)\}_j = \ln(r) \cdot g_r = \ln(g_r) < 1$. Consequently $\ln(r) \cdot \exp^{J-1}(g_r + 1) < \text{const} < 1$ for sufficiently large $J$'s and $(\frac{d}{dt} \exp^{M}(t))' \big|_{t = \log_r(g_r + 1)}$ tends to 0 for $M \to \infty$. □

Claim 7. For any $r \in (c, b)$ there is an embedding of the Cantor binary set $\{0, 1\}^\mathbb{N}$ into $X_r$. (This provides (i) of the proposition.)
PROOF. Take \( K \in \mathbb{N} \) so large that \( (\exp^K_x(t))'|_{t=\log_x(g+1)} < 1 \) (see Claim 6) and \( \exp^{K-1}_x(g+1) + 1 < \log_x(g+1) \). The latter implies that for any sequence \( \{m_k\}_k \) of positive integers satisfying \( \&_k m_k \geq K \) we have

\[
\&_k m_\mathcal{L}_k [g_r + 1, \log_r(g + 1)] \subseteq [g_r + 1, \log_r(g + 1)].
\]

Evidently \( \{m_\mathcal{L}_k [g_r + 1]\}_k \) is increasing and in view of the above inclusion is also bounded. For any 0-1-sequence \( l = \{l_k\}_k \) let \( f(l) = \lim \{m_\mathcal{L}_k (g_r + 1)\}_k \), where \( m_k = K + l_k \). It remains to check that \( f \) is continuous and injective.

**Injectivity.** Recall that \( f(\{0,1\}^\infty) \subseteq (g + 1, \log_r(g + 1)) \). Let \( k, 1 \in \{0,1\}^\infty \) coincide in the first \( n - 1 \) entries and \( k_n = 0, l_n = 1 \). Let \( k_q, l_q \) denote the \((q+1)\)-tails of \( k, l \) respectively i.e. \( (k_q)_p = k_{q+p}, (l_q)_p = l_{q+p} \) for \( q \in \mathbb{N} \). We have \( f(k_n), f(1_n) \in (g_r + 1, \log_r(g_r + 1)) \) and

\[
\begin{align*}
&f(k_{n-1}) \in 1 + \exp^K_x(g_r + 1, \log_r(g_r + 1)), \\
&f(1_{n-1}) \in 1 + \exp^{K+1}_x(g_r + 1, \log_r(g_r + 1))
\end{align*}
\]

with disjoint right-hand side sets. Since \( p_\mathcal{L}_n \), where \( p \) is \( K \)+(the common beginning of \( k, l \)), is a strictly increasing function and \( f(k) = p_\mathcal{L}_n(f(k_{n-1})) \), \( f(1) = p_\mathcal{L}_n(f(1_{n-1})) \) we have \( f(k) \neq f(l) \).

**Continuity.** Denote \( (\exp^K_x(t))'|_{t=\log_x(g+1)} = w, w < 1 \). Then

\[
\sup \{(d/dr)^m w | t \in (g_r + 1, \log_r(g_r + 1))\} < w^k
\]

whenever \( \&_k m_k \geq K \). Consequently if \( k, 1 \in \{0,1\}^\infty \) coincide in the first \( n \) entries then \( |f(k) - f(l)| < (\log_r(g_r + 1) - g_r - 1) \cdot wn \). \( \square \)

**Lemma 8.** Let \( f \in C^2(\mathbb{R}), f(0) = 0, f'(0) \in \mathbb{R}_+ - \{1\} \). For any \( x_0 \in \mathbb{R}_+ \) set \( x_k = f(x_{k-1}), k \geq 1 \). Then \( \forall s > 0: \exists q > 0: \) for any \( n \),

(i) \( x_0(1 - s) < x_n(f'(0))^{-n} < x_0(1 + s) \),
(ii) \( (x_0 - y_0)(1 - s) < (x_n - y_n)(f'(0))^{-n} < (x_0 - y_0)(1 + s) \),

whenever \( 0 < y_0 < x_0 < q \) for \( f'(0) < 1 \) or \( 0 < y_n < x_n < q \) for \( f'(0) > 1 \).

**Proof.** It suffices to consider only \( f'(0) < 1 \). We have \( x_k = f'(0)x_{k-1} + u_kx_{k-1}^2 \) with \( u_k = f''(a \text{ real } (0, x_{k-1})) \). So

\[
x_k/x_{k-1} = f'(0) \cdot (1 + u_k x_{k-1}/f'(0))
\]

and

\[
x_n/x_0 = (f'(0))^n \cdot \prod_{k=1}^n (1 + u_k x_{k-1}/f'(0)).
\]

We also have \( x_k - y_k = (x_{k-1} - y_{k-1})v_k \) with \( v_k = f'(a \text{ real } (y_{k-1}, x_{k-1})) = f'(0) + w_kx_{k-1} \) with \( w_k = (a \text{ real } (0, 1)) \cdot f''(a \text{ real } (0, x_{k-1})) \) and \( \{w_k\}_k \) is bounded. So

\[
(x_n - y_n)/(x_0 - y_0) = (f'(0))^n \cdot \prod_{k=1}^n (1 + w_k x_{k-1}/f'(0)).
\]

And since for sufficiently small \( x_0 \) we have \( \&_n x_n < x_0 \cdot t^n \) for any \( t > f'(0) \), both products \( \prod_{k=1}^n (1 + u_k x_{k-1}/f'(0)) \) and \( \prod_{k=1}^n (1 + w_k x_{k-1}/f'(0)) \) can be made arbitrarily close to 1 by appropriate choice of \( q \). \( \square \)
From now on assume that $\ln(\ln(g_b))/\ln(\ln(h_b))$ is irrational. Consider the set
$$W = \{\exp_b^m(\exp_b^n(1 + 1))| m, n \in \mathbb{N} \text{ and } \exp_b^n(1 + 1) > g_b\}.$$  
Clearly $W \subset (g_b, h_b)$. I will prove that $W$ is dense in $(g_b, h_b)$.

**Claim 9.** For any $s > 0$ there exists a sequence $\{w_k\}_{k \in \mathbb{N}} \subset W$ such that $w_k < w_{k+1} \& (h_b - w_k)/(h_b - w_{k+1}) < (1 + s)^5/(1 - s)^5$ for all $k$, and $\lim\{w_k\}_k = h_b$.

**PROOF.** $(b^t)^t = b^t \cdot \ln(b)$ which is $= \ln(g_b) < 1$ at $t = g_b$ and $= \ln(h_b) > 1$ at $t = h_b$. Let $s < 1$. Apply Lemma 8 to $b^t$ on $(0, g_b]$ (regarded as $g_b - t \to g_b - b^t$) to obtain $q_1$ and apply Lemma 8 to $b^t$ on $(g_b, h_b]$ (regarded as $h_b - t \to h_b - b^t$) to obtain $q_2$; let $q = \min(q_1, q_2)$. Take the minimal $l$ satisfying $\exp_b^l(1 + 1) > g_b - q \cdot (1 - s)^2$. Then for any $n$ we have

$$(1 - s)\ln(g_b))^n < \frac{g_b - \exp_b^{l+n}(1)}{g_b - \exp_b^l(1)} < (1 + s)(\ln(g_b))^n$$
and since $h_b = g_b + 1$ for any $m, n$ satisfying $\exp_b^m(\exp_b^n(1 + 1) > h_b - q$ we have

$$(1 - s)(\ln(h_b))^m < \frac{h_b - \exp_b^{l+n}(1)}{h_b - \exp_b^l(1)} < (1 + s)(\ln(h_b))^m;$$

brining together these inequalities we obtain

$$(1 - s)^2(\ln(h_b))^m(\ln(g_b))^n < \frac{h_b - \exp_b^{l+n}(1)}{g_b - \exp_b^l(1)} \left(1 + s\right)^2 < (1 + s)^2(\ln(h_b))^m(\ln(g_b))^n.$$  
Now, by the irrationality of $\ln(\ln(g_b))/\ln(\ln(h_b))$ one can choose $m, n$ so as to make $(\ln(h_b))^m(\ln(g_b))^n$ arbitrarily close to any positive real needed, in particular one can choose $(m_k, n_k)'s$ to approximate well from above each real of the form $(1 - s)^3/\left(1 + s\right)^3, k \in \mathbb{N}$. The corresponding $w_k = \exp_b^{m_k}(\exp_b^{l+n_k}(1 + 1))'s$ satisfy the requirements of the claim: $(1 + s)^{1-\varepsilon}/(1 - s)^{1-\varepsilon} < w_k/w_{k+1} < (1 + s)^5/(1 - s)^5$, where $w_i = h_b - w_i$. (Divide the inequalities for $w_i's$.)  

**Claim 10.** There exists $q > 0$ such that $W$ is dense in $(h_b - q, h_b)$.

**PROOF.** For $s = 1/2$ apply Lemma 8 to $b^t$ on $(0, g_b]$ to obtain $q_1, q_1 < 1$, and apply Lemma 8 to $b^t$ on $(g_b, h_b]$ to obtain $q_2, q_2 < 1$ (as in the beginning of Claim 9). Put $q = \frac{1}{2} \min(q_1, q_2)$. I will show that for each $u > 0$ there exists an increasing sequence $\{v_k\}_k \subset W$ such that $v_1 < h_b - q$ and $\&_k v_{k+1} - v_k < u$ with $\lim\{v_k\}_k = h_b$.

Apply Claim 9 with such an $S$ that $(w_{k+1} - w_k)/(h_b - w_{k+1}) < u/6q$ for each $k$. Let $n$ be the minimal positive integer such that $\exp_b^n(w_1) < h_b - q$. Then $\{v_k\}_k, v_k = \exp_b^n(w_k)$, satisfy the above requirements.  

Since $(g_b, g_b + 1) = (g_b, h_b) = \bigcup_{m=1}^{\infty} \exp_b^m(h_b - q, h_b)$ for any $q \in (0, 1)$ part (iii) of the proposition is completed.

**REFERENCES**


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