A NOTE ON THE EXISTENCE OF G-MAPS BETWEEN SPHERES

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Abstract. Let \( G \) be a finite group, and let \( V \) and \( W \) be finite-dimensional real orthogonal \( G \)-modules with \( V \supset W \), and with unit spheres \( S(V) \) and \( S(W) \) respectively. The purpose of this note is to give necessary sufficient conditions for the existence of a \( G \)-map \( f: S(V) \to S(W) \) in terms of the Burnside ring of \( G \) and its relationship with \( V \) and \( W \). Note that if \( W \) has a nonzero fixed point, such a \( G \)-map always exists, so for nontriviality, we assume this not the case.

Existence of \( G \)-maps. Let \( V \) be a finite-dimensional orthogonal \( G \)-module and let \( W \subset V \) be an invariant sub-\( G \)-module. Denote the unit spheres of \( V \) and \( W \) by \( S(V) \) and \( S(W) \) respectively. Here we obtain an algebraic criterion for the existence of a \( G \)-map \( f: S(V) \to S(W) \). Thus, for nontriviality, we assume \( W^G = \{0\} \).

The case \( V = W \) has been studied in [3], and we first recall pertinent facts. Let \( A(G) \) be the Burnside ring of \( G \). Thus, \( A(G) \) is the Grothendieck group of equivalence classes of finite \( G \)-sets with addition given by disjoint union. Its elements are thus represented by virtual \( G \)-sets, and \( A(G) \) is additively the free abelian group with basis \( \{G/H\} \), where \( H \) runs through representatives of conjugacy classes of subgroups of \( G \). The multiplicative structure is given by cartesian product. One has a natural isomorphism

\[ \Phi: A(G) \cong \omega, \]

where \( \omega \) denotes the zeroth equivariant stable stem. (See, for example, [2]. Roughly, \( \Phi \) is defined via the collapse map associated with a suitable embedding of a finite \( G \)-set in a large sphere \( S(V) \).) Denote by \( \phi(G) \) the set of conjugacy classes of subgroups of \( G \), and let

\[ d: A(G) \to \prod_{(H) \in \phi(G)} \mathbb{Z} = C \]

denote its integral closure. Thus \( d[s - t]_{(H)} = |s|^H - |t|^H \) for a virtual \( G \)-set \( s - t \).

It is well known that \( d \) is a monomorphism [1]. Denote by \( \Lambda(W) \) the monoid of (free) \( G \)-homotopy classes of \( G \)-maps \( S(W) \to S(W) \), and let \( \nu(W): \Lambda(W) \to A(G) \) denote the natural monoid homomorphisms obtained by suspending and applying \( \Phi^{-1} \). The results of [3] give a characterization of the image of \( \nu(W) \), which we now state. (The constructions there of \( G \)-maps \( S(W) \to S(W) \) representing suitable elements in \( A(G) \) are given in terms of appropriate tangent \( G \)-vector fields on \( S(W) \).)

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Proposition. An element \( a = [s - t] \in A(G) \) is in the image of \( \nu(W) \) iff the following conditions hold on \( s - t \).

(i) Recalling that \( W^G = \{0\} \), one requires that \( s - t \) be the form \( 1 + \sum_i n_i G/H_i \), where the \( H_i \) are isotropy subgroups of points in \( W - \{0\} \).

(ii) If \( H \) is an isotropy subgroup in \( W - \{0\} \) and \( \dim W^H = 1 \), then

\[
\deg(a)_{H} = \begin{cases} 
1 \text{ or } -1 & \text{if } NH \neq H; \\
0, 1 \text{ or } -1 & \text{if } NH = H.
\end{cases}
\]

One now has the following

Theorem. With \( V \) and \( W \) as above, there exists a \( G \)-map \( S(V) \to S(W) \) iff:

(a) For each \( H \subset G \), \( \dim V^H > 1 \) implies \( \dim W^H > 1 \).

(b) There exists \( a \in A(G) \) of the form \( 1 + \sum_i n_i G/H_i \) with each \( H_i \) an isotropy subgroup of \( W \) such that

(i) \( \deg(a)_{H} = 0 \) whenever \( \dim V^H > \dim W^H \);

(ii) if \( \dim V^H = 1 \), then

\[
\deg(a)_{H} = \begin{cases} 
1 \text{ or } -1 & \text{if } NH \neq H; \\
0, 1 \text{ or } -1 & \text{if } NH = H.
\end{cases}
\]

(\text{Note that (b) is equivalent to the following assertion:}

(b') There exists an element \( a \in \text{Im } \nu(W) \) with \( \deg(a)_{H} = 0 \) whenever \( \dim V^H > \dim W^H \).)

Proof. We first show that the conditions are necessary. Condition (a) is clearly necessary, while, given any \( G \)-map \( f: S(V) \to S(W) \), composing with the inclusion \( i: S(W) \to S(V) \) gives a \( G \)-map \( g \) on \( S(W) \) whose fixed set degrees are zero whenever \( \dim V^H > \dim W^H \), and we take \( a \) as \( \nu(W)(g) \).

Conversely, assume that conditions (a) and (b) hold. Choose \( a \in \text{Im } \nu(W) \) satisfying condition (b), and choose a \( G \)-map \( \rho: S(W) \to S(W) \) with \( \nu(W)(\rho) = a \). By condition (a), if \( H \subset G \), is such that \( \dim V^H > 1 \), then \( \dim W^H = 1 \) as well. This, together with condition (a) itself, permits one to define a \( G \)-map \( \lambda_0 \) from the zero skeleton of \( S(V) \) to \( S(W) \), with respect to some \( G \)-CW decomposition of \( S(V) \). Thus assume that we have constructed a \( G \)-map

\[
\lambda_n: S(V)^n \to S(W),
\]

where \( S(V)^n \) denotes the \( n \)-skeleton of \( S(V) \). The obstruction to extending \( \lambda_n \) over a typical \( (n + 1) \times G \)-cell of the form \( G/H \times D^{n+1} \) defines an element \( x \) of \( \pi_n(S(W)^H) \). We consider two cases. If \( n < \dim W^H - 1 \), then the obstruction vanishes for dimensional reasons, and one may extend over the given \( G \)-cell. If \( n \geq \dim W^H - 1 \), then one has \( \dim V^H > \dim W^H \). Let \( \lambda_n = \rho \circ \lambda_n \). Then, since now \( \deg(\rho^H) = 0 \), and since the obstruction to extending \( \lambda_n \) over the cell is given by \( \rho^H(x) = 0 \), the obstruction now vanishes. Continuing this process inductively now gives the desired result. \( \square \)

Remarks. Conditions (a) and (b) always hold in the following situation. Let \( G \) be any nonsolvable group. Then one has, by [1], a nontrivial idempotent \( e \) in \( A(G) \), and we may assume that \( \deg(e)_{\{1\}} = 0 \), where \( \{1\} \) denotes the trivial subgroup of \( G \).
If $H$ is a minimal subgroup for which $d(e)_H = 1$, then, if $\mathcal{F}: A(G) \to A(H)$ denotes the forgetful homomorphism (which assigns to any virtual $G$-set the associated $H$-set via restriction), one has, in $A(H)$, $d(\mathcal{F}(e))_K = 0$ for all proper subgroups $K \subset H$, while $d(\mathcal{F}(e))_H = 1$. Let $R$ be the reduced regular representation of $H$, let $W = R \oplus R$, and let $V = W \oplus W$. Then $\nu(W)$ contains all $H$-sets of the form $1 + \sum_i H/K_i$ with $K_i \subset H$ proper, whence it contains $\mathcal{F}(e)$. It follows from the theorem that there is an $H$-map $S(V) \to S(W)$. This in turn gives a $G$-map $S(iV) \to S(iW)$, where $i$ denotes induction.

The existence of such $G$-maps is by no means restricted to nonsolvable, or even to nonabelian groups; let $G = \mathbb{Z}/p \times \mathbb{Z}/q$, with $p$ and $q$ distinct primes. Choose integers $m$ and $n$ with $mp + nq = 1$, and let $V = \rho_p \oplus \rho_q \oplus \rho_{pq}$, $W = \rho_p \oplus \rho_q$, where $\rho_p$ is any one-dimensional irreducible complex $\mathbb{Z}/p$-module, regarded as a $(\mathbb{Z}/p \times \mathbb{Z}/q \equiv \mathbb{Z}/pq)$-module via projection, and similarly for $\rho_q$ and $\rho_{pq}$. Then $S(V)$ and $S(W)$ possess isomorphic fixed-sets by any nontrivial subgroup, and we may take $a = 1 - m\mathbb{Z}/p - n\mathbb{Z}/q$ as our element in $A(\mathbb{Z}/pq)$.

We state an easy consequence of the theorem.

**Corollary.** Let $W \subset V$ be any $G$-modules with $V^G = W^G$, and assume that if $H \subset G$ and $V^H \neq V^G$, then $W^H \neq W^G$. Denote the orthogonal complement of $V^G$ by $V(G)$, and similarly for $W(G)$. Then there exists a $G$-map $f: S(V) \to S(W)$ with fixed-set degree prime to $|G|$ iff (b) above holds with $V$ and $W$ there replaced by $V(G)$ and $W(G)$ respectively. (The condition on fixed sets by subgroups may be thought of as a mild “gap hypothesis,” and guarantees that (a) holds in this context.)

**Proof.** Note that if $V^G = W^G = 0$, then this is just a restatement of the theorem. Thus assume $\dim V^G \geq 1$. Conditions (a) and (b) are certainly sufficient; one may suspend any $G$-map $S^n \to S^n$, where $n = \dim V^G - 1$, with the unreduced suspension of a $G$-map $S(V(G)) \to S(W(G))$ to obtain a $G$-map of the desired degree. Conversely, given a $G$-map $f: S(V) \to S(W)$ with fixed-set degree prime to $|G|$, one has, for suitable $m$, $m \deg(f^G) \equiv 1 \mod |G|$ (and where we may take $m = \pm 1$ if $\dim V^G = 1$). This in turn gives an element $a \in A(G)$ satisfying condition (b) of the hypothesis of the theorem. Indeed, one obtains, by classical general position arguments, an element $a \in A(G)$ which represents the $G$-map $mf \circ$ (inclusion of $S(W)$ in $S(V)$). The crucial point here is that, since $V^G \neq 0$, general position arguments work, since we have a stationary basepoint to map into. Observing that any representing virtual $G$-set has orbit-types those of $W$, and that if $H \neq G$ is any isotropy subgroup occurring in $W$, one has $\dim W^H \neq 1$, one now sees that $a \in \nu(W(G))$ and satisfies the conditions (a) and (b) of the theorem.

**References**


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