ADDITIVE GROUPS OF T-RINGS
GEORGE V. WILSON

ABSTRACT. We build on a result of Bowshell and Schultz to give a complete characterization of the groups which occur as the additive groups of T-rings. This answers a question of Feigelstock.

In this paper, all groups are abelian. We write $R^+$ for the additive group of a ring $R$, $t(G)$ for the torsion subgroup of a group $G$, and $G_p$ for the $p$-primary component of $G$.

Fuchs [Fl, Problem 45] posed the problem of classifying those rings $R$ with the property that $R \cong \text{End}_\mathbb{Z}(R^+)$. Bowshell and Schultz [BS] found one class of rings with this property that they called T-rings. A unital ring $R$ is a T-ring if the map $m: R \otimes \mathbb{Z} R \to R$ induced by multiplication $a \otimes b \to ab$ is an isomorphism. [BS] characterized T-rings in terms of their additive groups. In order to state their result, we recall some basic facts about torsion-free groups. A torsion-free group has rank one if and only if it is isomorphic to a subgroup of the rational numbers. Rank one groups are completely classified by an invariant called type. A rank one group is the additive group of a unital ring if and only if its type is idempotent under a natural product. Readers unfamiliar with this material are referred to [F2, §85] for a thorough treatment.

We are now in a position to state the above-mentioned result.

**PROPOSITION 1 [BS].** The following are equivalent for a unital ring $R$:
(A) $R$ is a T-ring,
(B) (1) the quotient $R^+/t(R^+)$ is a rank one group of idempotent type and
(2) if $R^+_p$ is nonzero, it is cyclic and $R^+/t(R^+)$ is $p$-divisible.

From this, one can easily see that a torsion group is the additive group of a T-ring if and only if it is cyclic; see [BS, 1.4]. This led Feigelstock to ask if the conditions in B above are enough to guarantee that a nontorsion group is the additive group of a T-ring [Fg, Question 4.7.30]. By Proposition 1, it suffices to determine if such a group is the additive group of some unital ring. In fact, these conditions are not sufficient and we now determine the minor additional restriction needed to give a T-ring.

**PROPOSITION 2.** Let $G$ be an abelian group which satisfies the conditions in (B) above. Then either $G \cong t(G) \oplus G/t(G)$ or else $\bigoplus G_p \leq G \leq \prod G_p$.

**PROOF.** Since we assume that each nonzero $p$-component $G_p$ is cyclic, the $G_p$ are pure and bounded, hence they are summands [F2, 27.5]. Let $\pi_p: G \to G_p$ be a projection map and let $\alpha = \prod \pi_p: G \to \prod G_p$ be the product of these maps. Suppose that there is even one element $x$ with infinitely many of the $\pi_p x \neq 0$. In

Received by the editors September 30, 1985 and, in revised form, December 27, 1985.

©1987 American Mathematical Society
0002-9939/87 $1.00 + .25 per page

219
this case, we claim that \( \alpha \) is an injection. Let \( q: G \to G/t(G) \) be the quotient map, \( q(x) \neq 0 \). Take any \( g \in G \). If \( g \in t(G) \), then of course \( \alpha(g) \neq 0 \). Next suppose \( g \notin t(G) \). Since \( G/t(G) \) is assumed to be a rank one group, \( q(g) \) is a rational multiple of \( q(x) \), i.e. \( mq(x) = nq(g) \) for some integers \( m \) and \( n \). This implies that \( mx = ng + t \) for some \( t \in t(G) \). Since \( x \) has infinitely many nonzero projections \( \pi_p x \), so does \( mx \). Since \( t \) has only finitely many nonzero projections, \( ng \) must have infinitely many and so \( g \) does also. Thus, \( ag \neq 0 \) and \( \alpha \) is an injection.

Next suppose that there is no element with infinitely many projections. Then every element \( g \in G \) can be written \( g = k + t \), where \( k \in \ker \alpha \) and \( t \in t(G) \). Since \( \ker \alpha \cap t(G) = 0 \), \( G = \ker \alpha \oplus t(G) \).

We can now complete the picture of the additive groups of \( T \)-rings.

**Proposition 3.** A group \( G \) is the additive group of a \( T \)-ring if and only if

1. it satisfies the conditions of \( (B) \) in Proposition 1 and
2. there is some \( g \in G \) such that for every prime \( p \), \( \pi_p g \) generates \( G_p \).

**Proof.** Say \( G \approx R^+ \) for a \( T \)-ring \( R \). Proposition 1 shows that \( (B) \) holds. It is easy to see that for \( 1 \in R \), \( \pi_1 \) generates \( G_p \) for every \( p \).

Conversely, assume that \( G \) satisfies \( (1) \) and \( (2) \). By Proposition 2, \( G \approx t(G) \oplus G/t(G) \) or \( \bigoplus G_p \leq G \leq \prod G_p \). Suppose that \( G \approx t(G) \oplus X \), where \( X \) is a rank one, torsion-free group of idempotent type. Write the \( g \) given in condition \( (2) \) as \( g = t + x \), \( t \in (G) \), \( x \in X \). Since \( \pi_p(x) = 0 \) for all \( p \) and \( \pi_p(t) \neq 0 \) for only finitely many \( p \), \( \pi_p(g) \) is nonzero for only finitely many \( p \). Since \( \pi_p(g) \) generates each \( G_p \), only finitely many \( G_p \) are nonzero. Since each \( G_p \) is cyclic, \( t(G) \) is cyclic and carries a unital ring structure. Since \( X \) is rank one of idempotent type, it also has unital ring structure. Clearly, \( G \approx t(G) \oplus X \) carries the product ring structure.

Suppose \( \bigoplus G_p \leq G \leq \prod G_p \). Let \( q: G \to G/t(G) \) be the quotient map. Choose \( g \in G \) such that for all \( p \), \( \pi_p g \) generates \( G_p \) and so that \( q(g) \) has an idempotent height sequence. Give each \( G_p \) the ring structure that makes \( \pi_p g \) the unity of \( G_p \) and give \( \prod G_p \) the product ring structure. We claim that \( G \) is a subring. Clearly, if \( t \in \bigoplus G_p \) and \( x \) is any element of \( \prod G_p \), then \( xt \in \bigoplus G_p \leq G \), so we must show that the product of nontorsion elements of \( G \) is again in \( G \). Take \( x, y \in G \) and write \( jx = mg + t \), \( ky = ng + s \) with \( j, k, m, n \in \mathbb{Z} \), \( t, s \in t(G) \). Since \( q(g) \) has idempotent type and is divisible by \( j \) and \( k \), it is divisible by \( jk \). Choose \( z \in G \) with \( jkz = mng + u \), \( u \in t(G) \). With this choice, \( jk(z - xy) \in t(G) \), so \( v = z - xy \in t(G) \) and \( xy = z - v \in G \). We see that \( G \) is a unital subring of \( \prod G_p \) and so is a \( T \)-ring.

It is fairly easy to see that any group satisfying the conditions of \( (B) \) has a ring structure which makes the multiplication map \( G \circ G \to G \) an isomorphism. The extra condition in Proposition 3 is simply to insure that this is a unital ring structure.

**References**


Department of Mathematics, University of Georgia, Athens, Georgia 30602