ON AN EIGENVALUE PROBLEM OF AHMAD AND LAZER
FOR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In connection with a problem posed by S. Ahmad and A. C. Lazer, we show the existence of a class of nonselfadjoint eigenvalue problems related to the equation \( y^{(n)} + \lambda p(x)y = 0 \) for which the general eigenvalues comparison is not true. We use a comparison principle for the zeros of the corresponding Cauchy problem.

This paper provides a contribution to the understanding of a problem raised by S. Ahmad and A. C. Lazer [1] in connection with the comparison of the eigenvalues for some multi-point boundary value problems which are not selfadjoint.

One is given the equation

\[
L_n y + \lambda p(x)y = 0,
\]

where \( p(x) \) is a continuous function of constant sign on an interval \( I \), \( \lambda \) is a parameter, and \( L_n y \) is a linear differential disconjugate operator of order \( n \), that is, the only solution of \( L_n y = 0 \) with \( n \) zeros on \( I \) (counting multiplicity) is \( y \equiv 0 \).

Let us consider the eigenvalue problem given by equation (1) and the system of boundary conditions

\[
\begin{align*}
L_i y(a) &= 0, & i & \in \{i_1, \ldots, i_k\}, \\
L_j y(b) &= 0, & j & \in \{j_1, \ldots, j_{n-k}\},
\end{align*}
\]

where \( a, b \in I, 1 \leq k \leq n-1, L_i y, i = 0, \ldots, n-1, \) are the quasi-derivatives of \( y(x) \) (see [7]), and \( \{i_1, \ldots, i_k\}, \{j_1, \ldots, j_{n-k}\} \) are two arbitrary sets of indices from the set \( \{0, \ldots, n-1\} \).

Problems of this type have been studied extensively (cf. [2, 3, 5]). In particular, Elias [5] has shown that if \((-1)^{n-k}p_1(x) < 0\), then the eigenvalues of problems (1) and (2) are real and nonnegative and form a divergence sequence \( \{\lambda_m\}_{m \in \mathbb{N}} \).

Ahmad and Lazer [1] have considered a particular type of boundary condition (2), that is

\[
\begin{align*}
y(a) &= y'(a) = \cdots = y^{(k-1)}(a) = 0, \\
y(b) &= y'(b) = \cdots = y^{(n-k-1)}(b) = 0,
\end{align*}
\]

and showed that if we set \( p = p_i, \) where \( p_i, i = 1, 2, \) are two continuous functions, considering the corresponding sequence of eigenvalues \( \{\lambda_{i,m}\}_{m \in \mathbb{N}}, i = 1, 2, \) ordered by magnitude, then the condition

\[
(-1)^{n-k}p_2(x) \leq (-1)^{n-k}p_1(x) < 0
\]
implies that
\[ \lambda_{1,1} \lambda_{1,2} \cdots \lambda_{1,m} \geq \lambda_{2,1} \lambda_{2,2} \cdots \lambda_{2,m} \]
for every \( m \geq 1 \). In the same paper they have raised the question of studying when the condition (4) also implies
\[ \lambda_{1,m} \geq \lambda_{2,m} \quad \text{for every } m \geq 1; \]
an assertion that is true in the selfadjoint case, that is when the operator \( L \) is selfadjoint, \( n \) is even, and \( k = n/2 \).

This paper aims at pointing out a general class of eigenvalue problems (1), (2) for which the eigenvalues' comparison does not follow from condition (4).

In the following we consider the operator \( L_n y = y^{(n)} \) and the case for which only one condition is set at one of the end points \( a \) or \( b \), that is \( k = 1 \) or \( k = n - 1 \). Since for \( n = 2 \) the problem is selfadjoint, in the following we also suppose that \( n \geq 3 \). According to this assumption, the problem (1), (2) becomes
\[ y^{(n)} + \lambda p(x)y = 0, \]
(5)
\[ y^{(i_1)}(a) = \cdots = y^{(i_k)}(a) = 0, \]
(6)
\[ y^{(i_1)}(b) = \cdots = y^{(j_{n-k})}(b) = 0 \]
with \( k = 1 \) and \( (-1)^{n-1} p(x) < 0 \) or \( k = n - 1 \) and \( p(x) > 0 \).

We prove the following:

**Theorem 1.** Let \( p_1(x) \) be continuous on \([a, b]\) with \((-1)^{n-k} p_1(x) < 0\). For every \( m \geq 2 \) there exist \( p_2(x) \in C[a, b] \) such that (4) is satisfied but \( \lambda_{2,m} > \lambda_{1,m} \).

We obtain this theorem as a consequence of the following result regarding extremal points. The \( i \)th extremal point \( \theta_i(a) \) (cf. [6]), relative to the equation
\[ y^{(n)} + p(x)y = 0 \]
(7)
and system (6), is defined (when it exists) as the \( i \)th value of \( b \) in \((a, \infty)\) for which there exists a nontrivial solution of (7) which satisfies (6).

Let us suppose now that \( k = n - 1 \); in agreement with Butler and Erbe [3] we say that the system (6) is admissible if, having called \( s \) the unique index from 0, \( \ldots, n - 1 \) that does not belong to \( \{i_1, \ldots, i_{n-1}\} \), we have \( j_1 \leq s \). If we set \( p(x) = p_j(x) \), \( j = 1, 2 \), in (7), then the corresponding \( i \)th extremal point is indicated by \( \theta_{j,i} \).

**Theorem 2.** Let \( p_1(x) \) and \( m \) be given, where \( p_1(x) \) is continuous and positive on \([a, \infty)\), and \( m \geq 1 \) \([m \geq 2]\), and suppose that \( \theta_{1,m} \) exists. If system (6) is not admissible \([\text{admissible}]\) there exists \( p_2(x) \in C[a, \infty) \) such that \( p_2(x) = p_1(x) > 0 \), \( \theta_{2,m} \) exists, and \( \theta_{2,i} > \theta_{1,i} \) for \( 1 \leq i \leq m \) \([2 \leq i \leq m]\). We remark that if (6) is admissible, then \( \theta_{2,1} \leq \theta_{1,1} \) (see [2, Theorem 2]).

**A comparison principle.** Let us begin by stating some notation which we use in the following.

We say that a nonnull vector of \( \mathbb{R}^n \), \( \eta = (\eta_1, \ldots, \eta_n) \), has the \( D \)-property if there exist no three indices \( i, j, k \) such that \( i < j < k \) and \( \eta_i \eta_j < 0 \), \( \eta_j \eta_k < 0 \).
We say that \( \eta \) has the strictly \( D \)-property if there exists an index \( i \) such that the real numbers \( \eta_1, \ldots, \eta_{i-1}, (-1)\eta_{i+1}, \ldots, (-1)\eta_n \) are all different from zero and have the same sign.

If \( \eta \) has the \( D \)-property, we denote by \( r(\eta) \) the greatest index such that \( \eta_{r(\eta)} \neq 0 \) and \( \eta_{r(\eta)} \eta_i > 0 \) for every \( i \leq r(\eta) \).

Now let \( y(x) \) be the solution of the Cauchy problem

\[
y^{(n)} + p(x)y = 0, \quad y^{(i)}(\xi) = \eta_{i+1}, \quad i = 0, 1, \ldots, n-1,
\]

with \( \xi \in \mathbb{R} \) and \( p(x) > 0 \).

If \( \eta_i = \delta_{i,i} \) for a given \( i, 1 \leq i \leq n \), the solution of (8) will be denoted by \( u_i(x) \). These solutions will also be called the principal solution of (8).

Every solution \( y(x) \) of (8) can have only isolated zeros in a compact interval \([\xi, c], c > \xi \) (cf. [4, Proposition 1, p. 81]). Also, for the form of the equation and Rolle's theorem the quasi-derivatives of \( y(x) \), that is \( y(x), y'(x), \ldots, y^{(n-1)}(x) \), can have only isolated zeros.

Let \( z_1 < \cdots < z_m \) be the ordered set of the zeros (eventually empty) of the quasi-derivatives of \( y(x) \) in an interval \([\xi, c]\) and let \( Y(x) \) be the vector \( (y(x), y'(x), \ldots, y^{(n-1)}(x)) \).

**Lemma 1.** If \( \eta \) has the \( D \)-property, then \( Y(x) \) has the strictly \( D \)-property for \( x > \xi \). Moreover \( y^{(j)}(x), 0 \leq j \leq n-1 \), vanishes at the point \( z_i, i \geq 1 \), if and only if \( j \equiv (r(\eta) - i) \mod n \).

**Proof.** It is not restrictive to assume \( \eta_{r(\eta)} > 0 \). Let \( \varepsilon > 0 \) such that \( 0 < \varepsilon < z_1 - \xi \). In the interval \((\xi, \xi + \varepsilon)\) the functions \( y^{(i)}(x) \) are all positive and increasing for \( i = 0, \ldots, r(\eta) - 2 \); all negative and decreasing for \( i = r(\eta), r(\eta) + 1, \ldots, n - 1 \), while \( y^{(r(\eta) - 1)}(x) \) is positive and decreasing. This situation can change only if \( y^{(r(\eta) - 1)}(x) \) vanishes. Therefore, if \( z_1 \) exists, it must be a zero of \( y^{(r(\eta) - 1)}(x) \), moreover only this quasi-derivative of \( y(x) \) vanishes at this point, and \( Y(x) \) has the strictly \( D \)-property for \( x \in (\xi, z_1] \). This argument can be repeated in every interval \((z_i, z_{i+1}], i = 1, \ldots, m - 1\), proving the lemma.

Let \( j, 0 \leq j \leq n - 1 \), be a fixed index and consider the functions \( u_1^{(j)}(x), u_2^{(j)}(x), \ldots, u_n^{(j)}(x) \). Denote by \( w_1 < \cdots < w_m \) the ordered set (possibly empty) of the zeros of these functions on an interval \((\xi, c]\).

**Lemma 2.** \( u_l^{(j)}(x), 1 \leq l \leq n \), vanishes at the point \( w_i, i \geq 1 \), if and only if \( l \equiv (i + j) \mod n \).

**Proof.** The functions \( u_1^{(j)}(x), u_2^{(j)}(x), l_1 \neq l_2 \), cannot have a common zero \( w_l \) on \((\xi, c]\), otherwise there is a nontrivial linear combination \( v(x) \) of them with two quasi-derivatives which vanish at \( w_l \); since the vector \((v(\xi), v'(\xi), \ldots, v^{(n-1)}(\xi))\) has the \( D \)-property, this is in contradiction to Lemma 1. Moreover between two zeros of \( u_l^{(j)}(x) \) there is a zero of every function \( u_l^{(j)}(x), l \neq l_1 \); otherwise there exists (see [4, Lemma 1, p. 4]) a nontrivial linear combination of two principal solutions with two quasi-derivatives which vanish at a point \( x_0 > \xi \), again in contradiction to Lemma 1.

Since \( u_{j+1}^{(j)}(\xi) = 1 \) and \( u_l^{(j)}(\xi) = 0 \) for every \( l \neq j + 1 \), from the preceding observations it follows that if \( w_1 \) exists, it must be a zero of \( u_{j+1}^{(j)}(x) \).
Suppose now that the lemma is true for \( w_1, \ldots, w_i \), but not for \( w_{i+1} \). This means that \( w_i \) is a zero of \( u_i^{(j)}(x) \) and, if \( l < n \) \( [l = n] \), that \( w_{i+1} \) is a zero of \( u_i^{(j)}(x) \), with \( l_1 \neq l + 1 \) \( [l_1 = 1] \). Since all zeros \( w_t, t \leq i \), are simple, it follows that

\[
 u_i^{(j)}(w_{i+1})u_i^{(j)}(w_{i+1}) < 0 \quad [u_i^{(j)}(w_{i+1})u_i^{(j)}(w_{i+1}) > 0].
\]

So there exists \( \alpha > 0 \) \( [\alpha < 0] \) for which

\[
 v_1(x) = u_i(x) + \alpha u_i(x) \quad [v_i(x) = u_i(x) + \alpha u_i(x)]
\]

is such that \( v_i^{(j)}(w_{i+1}) = 0 \). As \( u_i^{(j)}(w_{i+1}) = 0 \), there exists a nontrivial linear combination \( v_2(x) \) of \( v_i(x) \) and \( u_i(x) \) which has two quasi-derivatives which vanish at \( w_{i+1} \), but the vector \( (v_2(\xi), \ldots, v_2^{(n-1)}(\xi)) \) has the \( D \)-property and this contradicts Lemma 1.

The following proposition gives us a criterion to compare the zeros of two solutions of the Cauchy problem (8).

**Proposition.** Suppose that \( u_i^{(j)}(x) \), \( j + 1 \leq l \), has \( m \) zeros, \( w_1 < \cdots < w_m \), on \((\xi, c]\). If \( \eta \) is a vector with the \( D \)-property such that \( j + 1 \leq r(\eta) \leq l \) and \( \eta_t \neq 0 \) for at least one index \( i \neq l \), then the \( j \)-derivative of the solution \( y(x) \) of (8) has at least \( m \) zeros \( z_1 < \cdots < z_m \) on \((\xi, w_m]\) and \( z_i < w_i \) for every \( i \). Moreover if \( l = r(\eta) \), \( y^{(j)}(x) \) has exactly \( m \) zeros on \((\xi, w_m]\).

**Proof.** It is not restrictive to assume \( \eta_{r(\eta)} > 0 \), so that \( \eta_i \geq 0 \) for \( 1 \leq i \leq r(\eta) \), \( \eta_i \leq 0 \) for \( r(\eta) + 1 \leq i \leq n \). Suppose first that \( l = r(\eta) \). From Lemma 2 it follows that at the point \( w_i \) we have for all the indices \( t \neq l \), either \( \eta_t = 0 \) or \( \text{sgn}[\eta_t u_t^{(j)}(w_i)] = (-1)^t \). Since \( \eta_t \neq 0 \) for at least an index \( t \neq l \), from the relation \( y^{(j)}(x) = \sum_{i=1}^{n} \eta_i u_i^{(j)}(x) \) and by continuity it follows that \( y^{(j)}(x) \) has a zero in every interval \((w_i, w_{i+1})\), \( i = 1, \ldots, m - 1 \). But \( r(\eta) \geq j + 1 \) so that \( y^{(j)}(x) > 0 \) for \( \xi < x < \xi + \varepsilon \) and \( \varepsilon \) sufficiently small; this implies that \( y^{(j)}(x) \) must have a zero also in the interval \((\xi, w_1]\). If \( y^{(j)}(x) \) has two zeros in an interval \((w_i, w_{i+1})\) or \((\xi, w_1]\), then it is possible to consider a linear combination \( v(x) \) of \( y(x) \) and \( u_i(x) \) which has two quasi-derivatives which vanish at a point \( x_0 > \xi \). Since \( r(\eta) = l \), the initial conditions of \( v(x) \) determine a vector with the \( D \)-property and this contradicts Lemma 1.

If \( l > r(\eta) \), then by Lemma 2 \( u_{r(\eta)}^{(j)}(x) \) has \( m \) zeros \( w'_1 < w'_2 < \cdots < w'_m \) on \((\xi, w_m]\) and \( w'_i < w_i \) for every \( i \). Now if \( \eta_i \neq 0 \) only for \( i = r(\eta) \), then the proof is trivial; otherwise the conclusion follows from the case \( l = r(\eta) \).

We consider now the particular case of problem (8) for which \( \xi = 0 \) and \( p(x) \) is constant, that is \( p(x) = k^n \), \( k > 0 \). The problem becomes

\[
y^{(n)} + k^n y = 0, \quad y^{(i)}(0) = \eta_i + 1, \quad i = 0, 1, \ldots, n - 1.
\]

Since in this case we are interested in the dependence of \( k \), we indicate the solution of (9) with \( y(x, k) \) and the principal solutions with \( u_l(x, k) \), \( 1 \leq l \leq n \).

For every \( k > 0 \) the principal solutions are oscillatory (see [6, Remark, p. 188]). If \( \eta \) is a vector with the \( D \)-property, then from the Proposition the solution of (9) is also oscillatory for every \( k \). Then it is possible to consider the function \( h(k) \) which associates the abscissa of the first zero of \( y(x, k) \) in the interval \((0, +\infty)\) to \( k \).

**Lemma 3.** Let \( \eta \) be a vector with the \( D \)-property and \( y(x, k) \) be the solution of (9). Then

\[
\lim_{k \to +\infty} kh(k) = M > 0
\]
and
\[ \lim_{k \to +\infty} \frac{\partial^{(i)} y}{\partial x^{(i)}} (h(k), k) / \partial^{(i-1)} y}{\partial x^{(i-1)}} (h(k), k) = +\infty \]
for every  \( i \) such that  \( 2 \leq i \leq n - 1 \).

**Proof.** From the relations
\[ y(x, k) = \sum_{i=1}^{n} \eta_i u_i(x, k), \]
\[ u_i(x, k) = k^{1-i} u_i(kx, 1), \quad i = 1, 2, \ldots, n, \]
if follows that
\[ \frac{\partial^{(j)} y}{\partial x^{(j)}} (h(k), k) = \sum_{i=1}^{n} \eta_i k^{1-i+j} \frac{\partial^{(j)} u_i}{\partial x^{(j)}} (kh(k), 1). \]
For our definition,  \( h(k) \) is the first positive zero  \( y(x, k) \), therefore  \( kh(k) \) is the first positive zero of  \( y(x/k, k) = \sum_{i=1}^{n} \eta_i k^{1-i} u_i(x, 1) \). Let  \( t \) be the first index such that  \( r_t \neq 0 \). For  \( k \to +\infty \),  \( kh(k) \) tends to the first positive zero  \( w_t \) of  \( u_t(x, 1) \). Since  \( (\partial^{(j)} u_t/\partial x^{(j)})(w_1, 1) \neq 0 \) for  \( j = 1, 2, \ldots, n-1 \) by Lemma 1, the proof of the lemma then follows by relation (11).

**Proof of Theorem 2.** Let system (6) be nonadmissible.

Let  \( s \) be the unique index which does not belong to  \( \{i_1, i_2, \ldots, i_{n-1}\} \); then  \( \theta_{1,i}(a) \),  \( l = 1, 2 \), is the  \( l \)-th zero of the  \( j \)-th derivative of the solution  \( u_{s+1}(x) \) of (8), where  \( p(x) = p_1(x) \) and  \( \xi = a \). Let  \( x_1 \) be the first zero greater than  \( a \) of  \( u_{s+1}(x) \). Since  \( j_1 > s \), from Lemma 1 it follows that  \( a < x_1 < \theta_{1,1}(a) \). We denote also by  \( u_*(x) \) the principal solution  \( u_n(x) \) of (8), where  \( p(x) = p_1(x) \) and  \( \xi = x_1 \), and by  \( \theta_i(x_1) \) the  \( i \)-th zero greater than  \( x_1 \) of  \( u_*(x) \).

Let us suppose first that  \( \theta_m(x_1) \) exists. By Lemma 1,  \( u^{(i)}_{s+1}(x_1) < 0 \) for  \( i = 1, \ldots, n-1 \). Applying the Proposition with  \( \xi = x_1 \) and  \( l = n \) it results that  \( \theta_i(x_1) > \theta_{1,i}(a) \) for  \( i = 1, \ldots, m \). Since the zeros  \( \theta_i(x_1) \) are simple, by the continuous dependence of the initial conditions and the Proposition there exists  \( \bar{x} < x_1 \) and  \( \delta > 0 \) such that for every vector  \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \), with  \( \gamma_n = 1 \) and  \( 0 \leq \gamma_i \leq \delta \) for  \( i = 1, \ldots, n-1 \), and for every  \( x_0 \in [\bar{x}, x_1] \) the  \( j \)-th derivative of the solution of the problem
\[ y^{(n)} + p_1(x)y = 0, \quad y^{(i)}(x_0) = \gamma_{i+1}, \quad i = 0, \ldots, n-1, \]
has exactly  \( m \) zeros  \( z_1 < \cdots < z_m \) in  \( (x_0, \theta_m(x_1)) \) and we have that
\[ \theta_{1,i}(a) < z_i, \quad i = 1, \ldots, m. \]
Now let  \( \eta \) be the vector whose components are  \( \eta_i = u^{(i-1)}_{s+1}(\bar{x}) \). By Lemma 3, there exists  \( k_0 \) such that  \( h(k_0) + \bar{x} < x_1 \),  \( k_0^n > \max\{p_1(x), x \in [a, x_1]\} \), and if  \( y(x, k) \) is the solution of (9) it follows that
\[ 0 < \frac{\partial^{(i)} y}{\partial x^{(i)}} (h(k_0), k_0) / \partial^{(n-1)} y}{\partial x^{(n-1)}} (h(k_0), k_0) < \delta, \quad i = 1, \ldots, n - 2. \]
Consider the function
\[ \tilde{p}(x) = \begin{cases} 
   p_1(x) & \text{for } a \leq x \leq \bar{x}, \\
   k_0^p & \text{for } \bar{x} < x \leq \bar{x} + h(k_0), \\
   p_1(x) & \text{for } x > \bar{x} + h(k_0).
\end{cases} \]

For Lemma 1 the \(j_1\)th derivative of the solution \(u_{s+1}(x)\) of (8) with \(p(x) = \tilde{p}(x)\) and \(\xi = \alpha\) does not vanish in \((a, \bar{x} + h(k_0))\); from (13) and (12) it follows then that the \(i\)th zero of \(u_{s+1}^{(j)}(x)\) is greater than \(\theta_{i,1}(a)\) for every \(i \leq m\). The existence of a continuous function \(p_2(x) \geq \tilde{p}(x)\) which verifies the theorem then follows by the fact that the zeros of \(u_{s+1}^{(j)}(x)\) are simple and from the classical result on differential equations.

Consider now the case for which \(\theta_m(x_1)\) does not exist. Since the principal solutions of (9) are oscillatory, from (10) and Rolle’s theorem it follows that the \(i\)th, \(i \geq 1\), zero of \(u_{s+1}^{(j)}(x, k)\) tends to zero for \(k \to +\infty\). By Lemma 1 the vector \(\eta_i\), whose components are \(\eta_i = u_{s+1}^{(i-1)}(\theta_{1,m}(a))\), \(i = 1, \ldots, n\), has the D-property, therefore for the Proposition also the \(i\)th zero of the \(j_1\)th derivative of the solution of (9) which correspond to this vector tends to zero for \(k \to +\infty\). So it is possible to consider a function \(p_1'(x)\) such that \(p_1'(x) \geq p_1(x)\), \(p_1'(x) = p_1(x)\) for \(a < x < \theta_{1,m}(a)\), and the point \(\theta_{2,i}(x)\) corresponding to the new function \(p_1'(x)\) exists. The proof of the theorem then follows from the preceding case.

Let (6) be admissible.

By Lemma 1 the first zero \(x_1\) of \(u_{s+1}(x)\) belongs to the interval \([\theta_{1,1}(a), \theta_{1,2}(a))\). Therefore if we proceed in the same way as in the case for which system (6) is not admissible, we can prove the existence of a function \(p_2(x) \geq p_1(x)\) such that \(\theta_{2,i}(a) > \theta_{1,i}(a)\) for \(2 \leq i \leq m\) and this completes the proof of the theorem.

**Proof of Theorem 1.** Suppose first that \(k = n - 1\).

The function \(p_1(x)\) can be considered to be defined on all of the interval \([a, +\infty)\) setting \(p_1(x) = p_1(b)\) for \(x > b\). If system (6) is admissible, then \(\lambda_{1,1} > 0\) (see [5, Corollary 3]). Moreover \(\lambda_{1,m}\) is the \(m\)th eigenvalue of problem (5), (6), where \(p(x) = p_1(x)\), if and only if \(b\) is the \(m\)th extremal point relative to equation \(y^{(n)} + \lambda_{1,m}p_1(x)y = 0\) and system (6) (see [5, Theorem 3]). By Theorem 2 there exists \(p_2(x) \geq p_1(x)\) such that the \(m\)th \((m \geq 2)\) extremal point relative to the equation \(y^{(n)} + \lambda_{1,m}p_2(x)y = 0\) and system (6) is greater than \(b\). Since the positive eigenvalues of (1), (2) are decreasing functions of the point \(b\) (see [6, Corollary 5]), the \(m\)th eigenvalue \(\lambda_m\) of problem (5), (6), where \(p(x) = \lambda_{1,m}p_2(x)\), is greater than 1. Therefore \(\lambda_m = \lambda_{2,m}/\lambda_{1,1} > 1\) and then \(\lambda_{2,m} > \lambda_{1,m}\).

If the system (6) is not admissible, then \(\lambda_{1,1} = 0\) and \(\lambda_{1,m} > 0\) for \(m \geq 2\); therefore we can prove the theorem as in the preceding case using Theorem 2.

Suppose now that \(k = 1\).

We remark that \(y(x)\) is a solution of problem (5), (6) if and only if the function \(z(x) = y(b + a - x)\) is a solution of problem
\[
\begin{align*}
   z^{(n)} + (-1)^n\lambda p(b + a - x)z &= 0, \\
   z^{(j_1)}(a) &= \cdots = z^{(j_{n-k})}(a) = 0, \\
   z^{(i_1)}(b) &= \cdots = z^{(i_k)}(b) = 0.
\end{align*}
\]
Therefore the eigenvalues of problem (5), (6) are the same as the eigenvalues of problem (14), (15). It follows that the case $k = 1$ can be reduced to the case $k = n - 1$, and this completes the proof of the theorem.

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