JENSEN’S INEQUALITY FOR POSITIVE CONTRACTIONS
ON OPERATOR ALGEBRAS

DÉNES PETZ

ABSTRACT. Let \( \tau \) be a normal semifinite trace on a von Neumann algebra, and let \( f \) be a continuous convex function on the interval \([0, \infty)\) with \( f(0) = 0 \). For a positive element \( a \) of the algebra and a positive contraction \( \alpha \) on the algebra, the following inequality is obtained:

\[
\tau(f(\alpha(a))) \leq \tau(\alpha(f(a))).
\]

Versions of Jensen’s inequality play an important role in several parts of mathematics and applications. One of the standard tools in quantum statistical mechanics is the Peierls-Bogoliubov inequality

\[
\text{tr} f(A) \geq \sum_i f(\langle A \eta_i, \eta_i \rangle),
\]

where \( f \) is a convex function on the real line, \( A \) is a selfadjoint operator, and \( \{\eta_i\} \) is an orthonormal set (see [11], for example). Slightly more general, if \( \{P_i\} \) is a pairwise orthogonal family of projections and we set \( \alpha(A) = \sum_i P_i A P_i \) for a selfadjoint operator \( A \), then

\[
\text{tr} \alpha(f(A)) \geq \text{tr} f(\alpha(A))
\]

(see [2, 6, and 9]). Going further in this direction Brown and Kosaki [3] have recently proved that

\[
\tau(f(v^*av)) \leq \tau(v^*f(a)v),
\]

where \( \tau \) is a faithful normal semifinite trace on a von Neumann algebra, \( v \) is a contraction, and \( a \) is a positive element of the algebra. (In fact, \( a \) may be unbounded and affiliated with the algebra.) In the present paper we obtain a similar inequality for an arbitrary positive contraction \( \alpha \) on the algebra and show that

\[
\tau(f(\alpha(a))) \leq \tau(\alpha(f(a))).
\]

Here \( f \) is supposed to be a continuous convex function with \( f(0) = 0 \). (Note that \( \tau \) holds without \( \tau \) if \( f \) is operator convex; cf. Theorem IV.1 in [1].) The inequality \( (*) \) supports the experience that inequalities involving operators are more simply treatable provided both sides are inside of a trace (see [3 and 8] for further evidences).

The idea of our proof is to approximate the operators \( a \) and \( \alpha(a) \) by diagonal ones by means of their spectral resolution and to apply the classical Jensen’s inequality in

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273

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its discrete form. Comparing with [3] this way is more direct. (The main technical tool in [3] is the spectral dominance which is intimately related to the generalized s-numbers. See [3 Remark 5; 5, and 8].)

Our basic references on operator algebras are [4 and 10]. We recall that if $\tau$ is a normal semifinite trace on a von Neumann algebra $M$, then $\tau$ is weak* lower semicontinuous and admits an extension to a linear functional on the ideal linearly spanned by the set $\{a \in M_+: \tau(a) < +\infty\}$ (see [4, Chapter I, §6]. Let $h$ be a selfadjoint operator with Jordan decomposition $h_1 - h_2$. Following [3] we say that $\tau(h)$ is defined if $\tau(h_1) < +\infty$ or $\tau(h_2) < +\infty$. In this case, we set $\tau(h) = \tau(h_1) - \tau(h_2)$. If both $\tau(h)$ and $\tau(h')$ are defined and $\tau(h) + \tau(h')$ is well defined (i.e., does not have the form $\infty - \infty$), then $\tau(h + h') = \tau(h) + \tau(h')$ (cf. Lemma 9 in [3]).

**Theorem A.** Let $M$ and $N$ be von Neumann algebras and $f: \mathbb{R}^+ \to \mathbb{R}$ be a continuous convex function with $f(0) = 0$. Assume that $\tau$ is a normal semifinite trace on $M$ and $\alpha: N \to M$ is a positive contraction. Then for every $a \in N_+$ the inequality

\[
(\tau f(\alpha(a))) \leq \tau(\alpha(f(a)))
\]

holds provided that both sides are defined.

**Proof.** First we show that

\[
\tau(pf(pa(a)p)p) \leq \tau(p\alpha(f(a))p)
\]

for any projection $p$ in $M$ with $\tau(p) < +\infty$ and for any $a \in N_+$. (Note that in this case both sides of (1) are defined and finite.) Take an arbitrary $\varepsilon > 0$. We approximate $p\alpha(a)p$ by a diagonal operator $\sum_{i=1}^n \mu_i p_i$ such that $p_i$ is a spectral projection of $p\alpha(a)p$ for $1 \leq i \leq n$, $\sum_{i=1}^n p_i = p$,

\[
\tau(pf(pa(a)p)p) \leq \sum_{i=1}^n f(\mu_i)\tau(p_i) + \varepsilon,
\]

and

\[
\mu_i = \begin{cases} 
\tau(\alpha(a)p_i)/\tau(p_i) & \text{when } \tau(p_i) > 0, \\
0 & \text{otherwise.}
\end{cases}
\]

We choose $\delta > 0$ such that $|t_1 - t_2| \leq \delta$ and $0 \leq t_1 \leq \|a\|$ imply $|f(t_1) - f(t_2)| \leq \varepsilon/\tau(p)$. There exists a diagonal operator $b = \sum_{j=1}^k \nu_j q_j$ such that $\|a - b\| \leq \delta$, $\sum_{j=1}^k q_j = I$, and

\[
|\tau(p\alpha(f(a))p) - \tau(p\alpha(f(b))p)| \leq \varepsilon.
\]

By simple calculation we have

\[
f(\mu_i) = f \left( \sum_{j=1}^k \tau(\alpha(aq_j)p_i)/\tau(p_i) \right)
\]

\[
= f \left( \sum_{j=1}^k \nu_j \tau(\alpha(q_j)p_i)/\tau(p_i) + \tau(\alpha(a - b)p_i)/\tau(p_i) \right)
\]
when \( \tau(p_i) > 0 \) and \( 1 \leq i \leq n \). Here
\[
|\tau(\alpha(a - b)p_i)/\tau(p_i)| \leq \|\alpha(a - b)\| \leq \delta
\]
and so
\[
f(\mu_i) \leq f\left( \sum_{j=1}^{k} \nu_j \tau(\alpha(q_j)p_i)/\tau(p_i) \right) + \varepsilon/\tau(p).
\]
Now apply Jensen’s inequality
\[
f(\mu_i) \leq \sum_{j=1}^{k} f(\nu_j) \tau(\alpha(q_j)p_i)/\tau(p_i) + \varepsilon/\tau(p).
\]
Consequently
\[
\sum_{i=1}^{n} f(\mu_i)\tau(p_i) \leq \sum_{j=1}^{k} f(\nu_j) \tau\left( \alpha(q_j) \left( \sum_{i=1}^{n} \tau(p_i) \right) \right) + \varepsilon
\]
\[
= \tau(p\alpha(f(b))p) + \varepsilon.
\]
Finally we arrive at the estimate
\[
\tau(pf(p\alpha(a)p)p) \leq \tau(p\alpha(f(a))p) + 3\varepsilon
\]
and since \( \varepsilon \) was arbitrary (1) is proved.

When \( \tau \) is finite the proof is complete. If not, then a supplementary argument is needed. There is \( c \in \mathbb{R}^+ \cup \{+\infty\} \) such that \( f \geq 0 \) on \( (c, +\infty) = I_1 \) and \( f \leq 0 \) on \( [0, c] = I_2 \). Let \( q_i \) be the spectral projection of \( \alpha(a) \) corresponding to \( I_i \) \((i = 1, 2)\). Since \( q_1 f(\alpha(a)) - q_2 (-f(\alpha(a))) \) is the Jordan decomposition of \( f(\alpha(a)) \) we have
\[
\tau(f(\alpha(a))) = \tau(q_1 f(\alpha(a))) + \tau(q_2 f(\alpha(a))).
\]
Let \( d_1 - d_2 \) be the Jordan decomposition of \( \alpha(f(a)) \). Since \( \tau(d_1) < \infty \) or \( \tau(d_2) < \infty \) the difference \( \tau(pd_1) - \tau(pd_2) \) is well defined and equal to \( \tau(p\alpha(f(a))) \) for any projection \( p \).

By the semifiniteness of \( \tau \) there exists an increasing net \( (p^s_i)_{s \in S} \) of projections in \( M \) such that \( p^s_i \rightarrow q_i \) \((i = 1, 2)\) and \( \tau(Ip^s_i) < +\infty \) \((i = 1, 2 \text{ and } s \in S)\). For any \( s \in S \) we have proved that
\[
\tau(p^s_i f(p^s_i \alpha(a)p^s_i)p^s_i) \leq \tau(p^s_i \alpha(f(a))p^s_i).
\]
Since \( \tau \) is weak* lower semicontinuous and
\[
p^s_i f(p^s_i \alpha(a)p^s_i)p^s_i \rightarrow q_i f(q_i \alpha(a)q_i) = q_i f(\alpha(a))
\]
in the strong operator topology \([10, II, 4.6]\) and hence in the weak* topology, we establish
\[
\tau(q_i f(\alpha(a))) \leq \liminf_s \tau(p^s_i f(p^s_i \alpha(a)p^s_i)p^s_i)
\]
\[
\leq \lim_s \tau(p^s_i d_1) - \lim_s \tau(p^s_i d_2) = \tau(q_i \alpha(f(a))).
\]
Adding these inequalities for \( i = 1 \) and \( i = 2 \) we complete the proof.
In the whole proof of the theorem we needed \( f(0) = 0 \) only when the discrete Jensen’s inequality was applied to a degenerate convex combination

\[
\sum_{j=1}^{k} \nu_j \frac{\tau(\alpha(q_j)p_i)}{\tau(p_i)}
\]

(that is, \( \sum_{j=1}^{k} \frac{\tau(\alpha(q_j)p_i)}{\tau(p_i)} \leq 1 \)). If we replace \( f(0) = 0 \) by the assumption that \( \alpha \) is unit preserving, then

\[
\sum_{j=1}^{k} \tau(\alpha(q_j)p_i)/\tau(p_i) = \tau\left(\alpha \left(\sum_{j=1}^{k} q_j\right)p_i \right) / \tau(p_i) = 1
\]

and the same arguments give another version of the previous theorem. (We recall that a positive unit preserving linear mapping is necessarily a contraction; see Lemma 1 in \([7]\), for example.)

**THEOREM B.** Let \( M \) and \( N \) be von Neumann algebras and \( f: \mathbb{R}^+ \rightarrow \mathbb{R} \) be a continuous convex function with \( f(0) = 0 \). Assume that \( \tau \) is a normal semifinite trace on \( M \) and \( \alpha: N \rightarrow M \) is a positive unit preserving mapping. Then for every \( a \in A^+ \) the inequality

\[
\tau(f(\alpha(a))) \leq \tau(\alpha(f(a)))
\]

holds whenever both sides are defined.

Theorems A and B have a simple consequence for positive contractions on \( C^* \)-algebras. Recall that a positive functional \( \tau \) on a \( C^* \)-algebra is called tracial if \( \tau(aa^*) = \tau(a*a) \) for every element \( a \).

**COROLLARY.** Let \( A \) and \( B \) be unital \( C^* \)-algebras and \( \alpha: A \rightarrow B \) be a positive contraction. Assume that \( f: \mathbb{R}^+ \rightarrow \mathbb{R} \) is a continuous convex function. If \( \tau \) is a tracial positive functional on \( B \) and

(i) \( \alpha \) is unit preserving, or
(ii) \( f(0) = 0 \),

then the inequality (*) holds for every positive \( a \in A \).

**PROOF.** The second adjoint of \( \alpha \) is an extension of \( \alpha \) and forms a positive contraction of \( A^{**} \) into \( B^{**} \). On the other hand, \( \tau \) admits an extension to a normal tracial positive functional of the enveloping von Neumann algebra \( B^{**} \). So one simply applies Theorems A and B for the enveloping von Neumann algebras \( A^{**} \) and \( B^{**} \).

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**REFERENCES**