RADON'S PROBLEM FOR SOME SURFACES IN $R^n$

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ABSTRACT. Radon's problem for a family of curves in $R^2$ has been generalized to a family of $(n - 1)$-dimensional surfaces in $R^n$. The problem is posed as a set of integral equations. Solutions to these equations are given for paraboloids and cardioids, and for these cases the null spaces and consistency conditions have been found.

In $R^n$ let $x = (x_1, x_2, \ldots, x_n)$ be a vector, let $\xi, \eta$ be unit vectors, and let $\cdot$ denote the scalar product. Let $r = |x|$, $x = r\xi$, and let $p$ be a nonnegative real number. For a fixed $p, \eta$ the expression

$$r^\alpha \cos(\alpha \cos^{-1}(\xi \cdot \eta)) = p^\alpha, \quad \alpha > 0,$$

represents an $(n - 1)$-dimensional surface which is symmetrical about $\eta$ and for which $r = p$ when $\xi = \eta$. Radon's problem is to determine a function $f(x)$ given the integrals of $f$ over the surfaces (1). This is a generalization of Radon's problem in which (1) represented a family of curves in $R^2$, which was discussed in [1, 2, 3]. In this two-dimensional problem $\alpha$ was assumed to be positive and the curves were called $\alpha$-curves, and for $\alpha$ negative we defined $\beta = -\alpha$ and referred to these curves as $\beta$-curves. The $\alpha$-curves and $\beta$-curves are intimately related since an $\alpha$-curve becomes the corresponding $\beta$-curve under inversion in the unit circle: $(r, \theta) \rightarrow (1/r, \theta)$. In what follows we shall first consider the surfaces (1) for $\alpha$ as stated, namely, $\alpha > 0$, and then in Appendix A list the important formulas for the $\beta$-surfaces with corresponding formula numbers since the treatments are so similar that the arguments need not be repeated.

We first establish the integral equations for $f$ when $f$ is expanded in spherical harmonics. For $\alpha = 1/2$ these equations reduce to an extension of an integral equation which was solved by Wimp [13], and which is discussed in Appendix B. For $\alpha = 1/2$, (1) represents a family of paraboloids. Their closest point to the origin is $x = p\eta$, and for $p = 0$ they degenerate to the straight line from the origin to infinity in the direction $-\eta$. For $\alpha = -1/2$ ($\beta = 1/2$), (1) represents a family of cardioids. Their greatest distance from the origin is $x = p\eta$ and $r \rightarrow 0$ as $\xi \rightarrow -\eta$. The solution in this case depends on a further modification of Wimp's result, also discussed in Appendix B.

In addition to the solutions of Radon's problem for these paraboloids and cardioids we give the null-spaces and consistency conditions for the corresponding Radon transforms.

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We restrict ourselves to \( n \geq 3 \) because the results for \( n = 2 \) are known [1, 2, 3]. For many but perhaps not all of the results given below the formulas for \( n = 2 \) may be obtained by using the formulas

\[
\lim_{\lambda \to 0} \Gamma(\lambda) C_\lambda^\lambda(\cos \theta) = (2/l) T_l(\cos \theta) = (2/l) \cos l\theta,
\]

where \( C_\lambda^\lambda \) is a Gegenbauer polynomial and \( T_l(x) \) is a Tchebycheff polynomial of the first kind [9].

Let \( f(x) \) be a smooth, rapidly decreasing function of \( x \) and let \( dx \) be an element of volume of \( \mathbb{R}^n \). The integral of \( f \) over the surface (1) for a particular \((p, \eta)\) will be denoted by \( \hat{f}(p, \eta) \) which will be called the Radon transform of \( f \). The case \( \alpha = 1 \) is the well-known case of integrals over planes discussed by Ludwig [8] and Deans [5], who gives numerous other references.

Let \( \delta(\cdot) \) be the Dirac delta-function and let \( h(\xi \cdot \eta) = \cos\{\cos^{-1}(\xi \cdot \eta)\} \). Then

\[
\alpha r^{\alpha-1} \delta(r^\alpha h(\xi \cdot \eta) - p^\alpha)
\]

represents a delta-function of unit weight concentrated on the surface (1). The factor \( \alpha r^{\alpha-1} \) ensures that the expression has unit weight (Papoulis [10]), that is to say the integral

\[
\hat{f}(p, \eta) = \int f(x) \alpha r^{\alpha-1} \delta(r^\alpha h(\xi \cdot \eta) - p^\alpha) \, dx
\]

is the integral over this surface in standard measure, the integral being taken over all space.

Suppose that \( f \) is now expanded in spherical harmonics [6]:

\[
f(x) = \sum_l f_{lm}(r) S_{lm}(\xi).
\]

Then \( f_{lm}(r) = r^l \times \) (an even function of \( r \), and the Radon transform of a typical term in (4) may be denoted by \( \hat{f}_{lm}(p, \eta) \) and written

\[
\hat{f}_{lm}(p, \eta) = \int f_{lm}(r) S_{lm}(\xi) \alpha r^{\alpha-1} \delta(r^\alpha h(\xi \cdot \eta) - p^\alpha) \, dx.
\]

In \( \mathbb{R}^n \), \( dx = r^{n-1} \, dr \, d\Omega_\xi = r^{2\lambda+1} \, dr \, d\Omega_\xi \), where \( \lambda = (n-2)/2 \) and \( d\Omega_\xi \) is an element of solid angle. Because \( n \geq 3 \), \( \lambda \geq 1/2 \). Thus (5) can be written

\[
\hat{f}_{lm}(p, \eta) = \alpha \int_{\Omega_\xi} S_{lm}(\xi) \, d\Omega_\xi \int_0^\infty f_{lm}(r) r^{2\lambda+\alpha} \delta(r^\alpha h - p^\alpha) \, dr.
\]

The \( r \)-integral is found to be \( p^{2\lambda+1} f_{lm}(p/h^{1/\alpha})/\alpha h^{(2\lambda+\alpha+1)/\alpha} \) so (6) becomes

\[
\hat{f}_{lm}(p, \eta) = p^{2\lambda+1} \int_{\Omega_\xi} f_{lm}(p/h^{1/\alpha}) S_{lm}(\xi) h^{-(2\lambda+\alpha+1)/\alpha} \, d\Omega_\xi.
\]

Since \( h = h(\xi \cdot \eta) \), the integral is of the form \( \int_{\Omega_\xi} G(\xi \cdot \eta) S_{lm}(\xi) \, d\Omega_\xi \) which, according to the Funk-Hecke theorem [6], is equal to \( \beta_l S_{lm}(\eta) \) where

\[
\beta_l = \frac{\omega_{n-1}}{C_\lambda^\lambda(1)} \int_{-1}^{+1} G(t) C_\lambda^\lambda(t)(1 - t^2)^{\lambda-1/2} \, dt,
\]
in which $\omega_{n-1}$ is the area of the unit sphere in $\mathbb{R}^{n-1}$ and $C_\lambda^\alpha$ are the Gegenbauer
or ultraspherical polynomials [6, 9]. Since $\beta_i$ is a function of $p$ only we may write

$$f_{im}(p, \eta) = \hat{f}_i(p)S_{im}(\eta),$$

where we have suppressed the $m$ in $\hat{f}_i(p)$, and where

$$\hat{f}_i(p) = p^{2\lambda+1} \frac{\omega_{n-1}}{C_\lambda^\alpha(1)} \int_{\cos(\pi/2\alpha)}^{\infty} f_i \left( \frac{p}{h^{1/\alpha}} \right)$$

$$\times \left[ h(t) \right]^{-(2\lambda+\alpha+1)/\alpha} C_\lambda^\alpha(t) \left( 1 - t^2 \right)^{-1/2} dt,$$

and the limits result from the specific form of $h(t)$, namely, $h(t) = \cos\{\alpha \cos^{-1} t\}$. On changing the integration variable from $t$ to $r = p/h^{1/\alpha}$, and on further putting $r^\alpha = s$ and $p^\alpha = q$ and defining

$$F_i(s) = (1/\alpha)f_i(s^{1/\alpha})s^{1/\alpha-1}, \quad \hat{F}_i(q) = \hat{f}_i(q^{1/\alpha}),$$

(10) becomes

$$\hat{F}_i(q) = \frac{\omega_{n-1}}{C_\lambda^\alpha(1)} \int_q^\infty F_i(s) \left[ \cos \left( \frac{1}{\alpha} \cos^{-1} \left( \frac{q}{s} \right) \right) \right]$$

$$\times \sin^{2\lambda} \left( \frac{q}{s} \right) \left( 1 - \left( \frac{q}{s} \right)^2 \right)^{-1/2} ds.$$

The solution of this integral equation for $F_i(s)$ will solve Radon’s problem for these surfaces. Quinto [11] has remarked that (12) is invertible for functions of compact support and that this implies a hole theorem (see below). No inversion formula
has been found except for the special cases $\alpha = 1/2, -1/2$ (paraboloids, cardioids) for which (12) can be reduced to a hypergeometric integral equation with a known solution. We proceed to discuss this solution and to give some properties of the $\hat{f}_i$.

For $\alpha = 1/2$, (12) becomes

$$\hat{F}_1(q) = \frac{2\omega_{n-1}}{C_\lambda^1(1)} \int_q^\infty f_1(s^2)s^4C_\lambda^1 \left( \cos \left( 2 \cos^{-1} \left( \frac{q}{s} \right) \right) \right)$$

$$\times \sin^{2\lambda} \left( \frac{q}{s} \right) \left( 1 - \left( \frac{q}{s} \right)^2 \right)^{-1/2} s ds.$$

Let $F(\cdot; \cdot; \cdot; \cdot)$ denote the ordinary hypergeometric function $2F_1$. Then it is known [9] that

$$C_\lambda^1(x) = \frac{\Gamma(l + 2\lambda)}{l!\Gamma(2\lambda)} F \left( -l, l + 2\lambda; \lambda + \frac{1}{2}; 1 - \frac{x}{2} \right).$$

If $\theta = \cos^{-1}(q/s)$, then use of the formulas for $\cos 2\theta$ and $\sin 2\theta$ enables (13) to be written

$$\hat{F}_1(q) = \frac{(2q)^2\Gamma(l + 2\lambda)2\omega_{n-1}}{l!\Gamma(2\lambda)C_\lambda^1(1)} \int_q^\infty f_1(s^2)s^2\lambda F \left( -l, l + 2\lambda; \lambda + \frac{1}{2}; 1 - \left( \frac{q}{s} \right)^2 \right)$$

$$\times \left( 1 - \left( \frac{q}{s} \right)^2 \right)^{\lambda-1/2} s ds,$$
and the further substitutions $s^2 = t$, $q^2 = r$ reduce (15) to

$$
\frac{\hat{f}_l(r)}{r^\lambda} = K_l \int_r^\infty f_l(t)t^{\lambda}F \left(-l, l + 2\lambda; \lambda + \frac{1}{2}; 1 - \frac{r}{t}\right) \left(1 - \frac{r}{t}\right)^{\lambda - 1/2} dt,
$$

where

$$
K_l = \frac{2^{2\lambda} \Gamma(l + 2\lambda)\omega_{n-1}}{\Gamma(2\lambda)C_l(1)}.
$$

For $l = 0$ the hypergeometric function is 1 and the resulting integral equation is easily solved by differentiation if $\lambda = 1/2, 3/2, \ldots$, or is reduced to Abel’s equation by differentiation when $\lambda = 1, 2, 3, \ldots$. In what follows $l > 0$.

If we make the identification $a = -l$, $b = l + 2\lambda$, and $c = \lambda + 1/2$, then (16) is precisely of the form of the modified Wimp equation (B3). The solution to (B3) is (B7), which contains an integer $m > c > 0$. In the case of planes [5, 8] and spheres through the origin [4] the derivative of $\hat{f}_l$ which appears in the solutions to the analog of (16) is the $(n - 1)$th derivative, so we choose $m = n - 1 = 2\lambda + 1$. The solution of (16) is then

$$
\hat{f}_l(x) = \frac{(-1)^{2\lambda + 1}}{(4\pi)^{\lambda+1/2}\Gamma(\lambda + 1/2)} \int_x^\infty (y - x)^{\lambda - 1/2}F \left(l, -(l + 2\lambda); \lambda + \frac{1}{2}; 1 - \frac{y}{x}\right)
\times \frac{d^{n-1}}{dy^{n-1}} \left(\frac{\hat{f}_l(y)}{y^\lambda}\right) dy.
$$

(18) illustrates the hole theorem: in order to find $f$ at $x_0$ it is only necessary to know $\hat{f}$ for $|y| > x_0$.

We now obtain for this paraboloidal Radon transform results about its null space, and the so-called consistency conditions, both of which are extensions to $\mathbb{R}^n$ of results in $\mathbb{R}^2$ [1, 2]. We rewrite (16) using the result [9]

$$
F \left(-l, l + 2\lambda; \lambda + \frac{1}{2}; 1 - x\right) = \frac{\Gamma(\lambda + 1/2)\Gamma(2\lambda + 2l)}{\Gamma(\lambda + l + 1/2)\Gamma(l + 2\lambda)} G_l \left(2\lambda, \lambda + \frac{1}{2}, x\right),
$$

where $G_l(p, q, x)$ is a shifted Jocobi polynomial. These polynomials form a complete set in $(0, 1)$ with a weight function $w(x) = x^{q-1}(1-x)^{p-q}$, $q > 0$, $p - q > -1$. The weight function for $G_l(2\lambda, \lambda + 1/2, x)$ is thus

$$
w_\lambda(x) = x^{\lambda - 1/2}(1 - x)^{\lambda - 1/2}.
$$

(16) becomes

$$
\frac{\hat{f}_l(r)}{r^{2\lambda+1}} = (4\pi)^{\lambda+2}(l+\lambda) \frac{\Gamma(l + \lambda)}{\Gamma(l + 2\lambda)} \int_0^1 f_l(t) t^\lambda G_l \left(2\lambda, \lambda + \frac{1}{2}, \frac{r}{t}\right) \left(1 - \frac{r}{t}\right)^{\lambda - 1/2} dt
$$

which, with a change of variable, may be written

$$
\frac{\hat{f}_l(r)}{r^{2\lambda+1}} = (4\pi)^{\lambda+2}(l+\lambda) \frac{\Gamma(l + \lambda)}{\Gamma(l + 2\lambda)} \int_0^1 f_l \left(\frac{r}{x}\right) x^{-2\lambda-3/2} G_l \left(2\lambda, \lambda + \frac{1}{2}, x\right) w_\lambda(x) dx.
$$
If \( f(t) = t^{-k} \), \( f_1(r/x)x^{-2\lambda - 3/2} = x^{k-2\lambda - 3/2}r^{-k} \), and because \( G_l \) is orthogonal to any polynomial in \( x \) of degree less than \( l \), the integral in (22) will vanish for \( k = l + 2\lambda - 1/2, l + 2\lambda - 3/2, \ldots, 2\lambda + 3/2 \). Thus the null space contains at least powers of \( t \) with these exponents.

To find the consistency conditions multiply (18) by \( r^\mu \) and integrate:

\[
\int_0^\infty \tilde{f}_1(r)r^{\mu - \lambda} \, dr = \text{const} \int_0^\infty r^\mu \, dr \int_0^\infty \tilde{f}_1(t)t^\lambda G_l \left( 2\lambda, \lambda + \frac{1}{2}, \frac{r}{t} \right)^{\lambda - 1/2} \, dt
\]

\[
= \text{const} \int_0^\infty f_1(t)t^{\lambda + \mu + 1} \, dt \int_0^1 x^{\mu - \lambda + 1/2} G_l \left( 2\lambda, \lambda + \frac{1}{2}, x \right) w_\lambda(x) \, dx.
\]

Again using the orthogonality of the \( G_l \), the \( x \) integral will vanish if \( \mu - \lambda = l - 3/2, l - 5/2, \ldots, -1/2 \). Thus the consistency conditions may be written

\[
\sum_{m=0}^{l-1} a_m \int_0^\infty \tilde{f}_1(r)r^{l -(3/2)-m} \, dr = 0,
\]

where the \( a_m \) are \( l \) arbitrary numbers, and (24) may be integrated by parts \( (n-1) \) times to give

\[
\sum_{k=0}^{l-1} \alpha_k \int_0^\infty \tilde{f}_1^{(n-1)}(r)r^{\lambda+(1/2)+k} \, dr,
\]

where the \( \alpha_k \) are other arbitrary constants.

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**Appendix A.** Equation numbers in this section such as \((An)\) are the analogs of \((n)\) in the main text.

The \( \beta \)-curves are defined by the equation

\[
(\beta p^\beta / r)\delta(\beta^2 - p^\beta h(\xi \cdot \eta)).
\]

On expanding \( f \) and \( \tilde{f} \) in spherical harmonics, applying the Funk-Hecke theorem, and defining \( F_l \) and \( \tilde{F}_l \) as in (11) we obtain the integral equation for the \( \beta \)-surfaces:

\[
\tilde{F}_l(q) = \frac{\sin^{2\lambda} \left( \frac{1}{\beta} \cos^{-1} \left( \frac{s}{q} \right) \right)}{s^{2\lambda / \beta}} \, ds.
\]

For \( \beta = 1/2 \) (cardioids) (A13) reduces to

\[
\mu^\lambda \tilde{f}_1(r) = K_l \int_0^r f_1(t)t^{2\lambda} F(-l, l + 2\lambda; \lambda + 1/2; 1 - t/r)(1 - t/r)^{\lambda - 1/2} \, dt.
\]
This is precisely of the form of the integral equation (B8) given in Appendix B and we can write down its solution from (B9):

\[
p^{5\lambda+1} f_t(p) = \frac{(-1)^{n}}{(4\pi)^{\lambda+1/2}\Gamma(\lambda + 1/2)} \times \int_{0}^{p} \frac{d^{n-1}}{dy^{n-1}} \left(y^{3\lambda} f_t(y)\right) \left(\frac{1}{y} - \frac{1}{y} - \frac{1}{2}\left(1 - \frac{p}{y}\right)\right)\times \left(\frac{p}{y} - 1\right)^{\lambda-1/2} y^{2\lambda} dy,
\]

where \(a, b, c,\) and \(m\) have been identified as they were for \(\alpha = 1/2.\) (A18) illustrates the "hole"-theorem for this case: in order to find \(f\) for some value of \(p\) it is only necessary to know \(\hat{f}\) for \(|y| \leq p.\)

In (A16), \(F(\cdot; \cdot; \cdot; \cdot)\) may again be expressed as a shifted Jacobi polynomial and use of the orthogonality of these polynomials yields the following information about the null-space of the transform and its consistency conditions:

(A22) the null-space of the transform contains \(f_t(r) = r^\gamma,\) where \(\gamma = l - 2\lambda - 3/2, l - 2\lambda - 5/2, \ldots, -2\lambda - 1/2,\)

(A23) \(\int_{0}^{\infty} \hat{f}_t(r) r^{-s} dr = 0\) if \(s = l - 1/2, l - 3/2, \ldots, 3/2.\)

Appendix B. An extension of a result due to Wimp [13].

Sneddon [12, p. 295, Problem 4-21] gives an identity which may be rewritten

(B1) \(\int_{s}^{t} (u - s)^{c-1}(t - u)^{m-c-1} F(a, b; c; 1 - \frac{u}{s}) F(-a, -b; m - c; 1 - \frac{u}{t}) u^{-m} du = \frac{\Gamma(c)\Gamma(m - c)}{\Gamma(m)} s^{c-m}(t - s)^{m-1},\)

where \(0 < c < m = \) integer. The substitution \(u = st/v\) in (B1) gives

(B2) \(\int_{s}^{t} (t - v)^{c-1}(v - s)^{m-c-1} F(a, b; c; 1 - \frac{t}{v}) F(-a, -b; m - c; 1 - \frac{s}{v}) dv = \frac{\Gamma(c)\Gamma(m - c)}{\Gamma(m)} (t - s)^{m-1}.\)

Consider the integral equation

(B3) \(H(y) = \int_{y}^{\infty} (x - y)^{c-1} F(a, b; c; 1 - y/x) G(x) dx,\)
where $G(x)$ is a smooth rapidly decreasing function of $x$. Multiply (B3) by $F(-a, -b; m - c; 1 - y/t)(y - c)^{m-c-1}y^{-m}$ and integrate $y$ from $t$ to $\infty$. Interchanging the order of integration on the R.H.S. yields

$$
\int_t^\infty H(y)F(-a, -b; m - c; 1 - y/t)(y - c)^{m-c-1}y^{-m} \, dy = (-1)^{m-1} \int_t^\infty G(x) \, dx,
$$

(B4)

$$
\times \int_x^t F(-a, -b; m - c; 1 - y/t)F(a, b; c; 1 - y/x)
$$

$$
\times (t - y)^{m-c-1}(y - x)^{c-1}y^{-m} \, dy.
$$

Use of (B1) gives

$$
\frac{t^c\Gamma(m)}{\Gamma(c)\Gamma(m - c)} \int_t^\infty H(y)F\left(-a, -b; m - c; 1 - \frac{y}{t}\right)(y - c)^{m-c-1}y^{-m} \, dy
$$

(B5)

$$
= \int_t^\infty x^{c-m}G(x)(x - t)^{m-1} \, dx = I,
$$

and since $d^mI/dt^m = (-1)^m(m - 1)!G(t)t^{c-m}$, we have

$$
G(t)t^{c-m} = \frac{(-1)^m}{\Gamma(c)\Gamma(m - c)} \frac{d^m}{dt^m} \int_t^\infty H(y)F\left(-a, -b; m - c; 1 - \frac{y}{t}\right)(y - t)^{m-c-1} \, dy.
$$

(B6)

The derivative may be taken under the integral sign to yield, as the solution of (B3),

$$
G(t) = \frac{(-1)^m}{\Gamma(c)\Gamma(m - c)} \int_t^\infty H^{(m)}(y)F\left(-a, -b; m - c; 1 - \frac{y}{t}\right)(y - t)^{m-c-1} \, dy,
$$

(B7)

where $H^{(m)}$ indicates the $m$th derivative of $H$.

Consider now the integral equation

$$
H(y) = \int_0^y (y - x)^{c-1}F(a, b; c; 1 - x/y)G(x) \, dx.
$$

(B8)

Using the same procedure and the identity (B2) we find the solution

$$
t^mG(t) = \frac{(-1)^{m-1}}{\Gamma(c)\Gamma(m - c)} \int_0^p \left\{y^{m-c}H(y)\right\}^{(m)}(p - y)^{m-c-1} \, dy
$$

$$
\times F\left(-a, -b; m - c; 1 - \frac{p}{y}\right) y^c \, dy.
$$

REFERENCES

11. E. T. Quinto, private communication.

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