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CONTINUOUS FUNCTIONS ON MULTIPOLAR SETS

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ABSTRACT. Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$ $(n > 1)$ be a product of n Brelot harmonic spaces each of which has a bounded potential, and let K be a compact subset of Ω. Then, K is an n-polar set with the property that every i-section $(1 \leqslant i < n)$ of K through any point in Ω is $(n - i)$ polar if and only if every positive continuous function on K can be extended to a continuous potential on Ω. Further, it has been shown that if f is a nonnegative continuus function on Ω with compact support, then MRf, the multireduced function of f over Ω, is also a continuous function on Ω.

1. Introduction. Let Ω_j for $j = 1, 2, \ldots, n$ be locally compact spaces with countable basis for the topology and be Brelot spaces [**5**]. The principal results of this paper (cf. Theorem 3 and Theorem 12) characterize certain exceptional compact sets K contained in the product space $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ in terms of extendibility of positive continuous functions on K, to a positive n-potential on the entire space. This is a natural generalization of the corresponding results in [**7**] in the case of a single harmonic space. A key result, independent of interest, needed in the proof of the main results in Theorem 5 shows that the multireduced function of a continuous function is continuous. The results of this paper formed a part of my Ph. D. thesis submitted to McGill University [**6**]. I would like to thank Professor Gowrisankaran for his help in preparation of this work.

We shall use the notation and results from [**3**, **4** and **8**] concerning n-harmonic, n-superharmonic functions, and n-potentials. We assume that there is a bounded n-superharmonic function on $\Omega_1 \times \cdots \times \Omega_n$. Notice that it is equivalent to the assumption that each Ω_j $(j = 1, \ldots, n)$ has a bounded potential. If constants are superharmonic, then this assumption trivially holds. Throughout this paper, unless it is explicitly mentioned, n is an integer $\geqslant 2$, and we denote $\Omega_1 \times \cdots \times \Omega_n$ by Ω.

In the course of proving our main results we need a number of results that are routine generalizations of similar ones in the single variable case. The proofs of some of them are not quite obvious, and can be found in [**6**].

DEFINITION 1. Let f be an extended real-valued function on Ω. Define $MRf(x) = \inf\{u(x)$: u is n-hyperharmonic, and $u \geqslant f$ on $\Omega\}$. For each subset E of Ω, let X_E be the characteristic function of E, and let $X_E \cdot f$ be the pointwise product function on Ω. The function $MR(X_E \cdot f)$ is called the multireduced function of f over E.

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Observe that $(MR(X_E \cdot f))^{\uparrow}$, the lower semicontinuous regularization (see [**1**]) of $MR(X_E \cdot f)$, is an n-hyperharmonic function on Ω.

Let us recall the definition of a section from [**8**].

DEFINITION 2. Let E be a subset of Ω, $n > 1$, and i an integer such that $1 \leqslant i \leqslant n-1$. Let $r_1, \ldots, r_i$ be integers with $1 \leqslant r_1 < r_2 < \cdots < r_i \leqslant n$. Let $t = (x_{s_1}, \ldots, x_{s_j})$ be a point in $\Omega_{s_1} \times \cdots \times \Omega_{s_j}$ $(1 \leqslant j \leqslant n)$ with $\{r_1, \ldots, r_i\} \subseteq \{s_1, \ldots, s_j\} \subseteq \{1, 2, \ldots, n\}$. Then the $(r_1, \ldots, r_i)$-section of E through t is defined as

$$z \text{ in } \prod_{k=1}^{n} \Omega_k, \quad (y_1, \ldots, y_n) \text{ is in } E, \text{whenever } y_k = x_k \text{ for } k \text{ in } \{r_1, \ldots, r_i\},$$
$$k \text{ not in } \{r_1, \ldots, r_i\}, \quad y_k = z_k \text{ for } k \text{ not in } \{r_1, \ldots, r_i\},$$

which would be denoted by $E[(r_1, \ldots, r_i), t]$. For x in Ω, an i-section of E through x is always denoted by $E[(r_1, \ldots, r_i), x]$ for some $(r_1, \ldots, r_i)$.

2. A characterization of a class of multipolar sets. The following theorem gives a necessary condition for a compact set to be n-polar. We recall that a set E is n-polar if there exists a positive n-superharmonic function which is identically infinity on E.

THEOREM 3. *Let K be a compact subset of Ω such that every positive continuous function on K can be uniformly approximated on K by positive n-superharmonic functions on Ω. Then,*

(1) *If K has more than one point, then K is an n-polar set.*

(2) *In addition, if point sets are polar in Ω_j for $j = 1, 2, \ldots, n$, then*

(a) *Every i-section of K through any point in Ω is $(n-i)$-polar for $i = 1, 2, \ldots, n-1$.*

(b) *Given z in Ω such that z not in K, there is a positive n-superharmonic function u on Ω such that $u(z) < \infty$ and $u(x) = \infty$ for all x in K.*

PROOF. (1) The proof of the fact that the hypothesis implies the n-polarity of K is very similar to the proof of Theorem 1 of [**7**].

(2.a) This part is proved by induction on n. Let us make the induction assumption that for all products of Brelot spaces $\Omega_1 \times \cdots \times \Omega_m$ with $m < n$, the hypothesis of the theorem implies the property (a).

Let us now consider a compact set K contained in Ω satisfying the hypothesis. Fix z in Ω, and i such that $1 \leqslant i \leqslant n-1$. By a suitable rearrangement, if necessary, we may consider the i-section $K[(1, 2, \ldots, i), z]$ to be the general i-section of K. This set is compact and contained in $\Omega_{i+1} \times \cdots \times \Omega_n$. Let us consider further the nontrivial case when $K[(1, 2, \ldots, i), z]$ contains more than one point. We shall show that this set is $(n-i)$-polar. By the induction hypothesis, it suffices to prove that an arbitrary positive continuous function f on $K[(1, 2, \ldots, i), z]$ can be uniformly approximated on $K[(1, 2, \ldots, i), z]$ by positive $(n-i)$-superharmonic functions.

Let us consider the copy $L[i, z]$ of the above compact set in K, viz., $L[i, z] = \{(z_1, \ldots, z_i, y_{i+1}, \ldots, y_n) \text{ in } K\}$. $L[i, z]$ is a compact subset of K, and f can be considered as a positive continuous function on this set. By the Tietze extension theorem, there exists a positive continuous function g on K which extends f. By hypothesis, for $\varepsilon > 0$, there exists a positive n-superharmonic function u on Ω such that $|u - g| < \varepsilon$ on K uniformly. Set

$$v(y_{i+1}, \dots, y_n) = u(z_1, \dots, z_i, y_{i+1}, \dots, y_n).$$

This function is $(n-i)$ superharmonic on $\Omega_{i+1} \times \cdots \times \Omega_n$, and clearly $|f - v| < \varepsilon$ uniformly on $K[(1,2,\dots,i), z]$. Hence, by the induction assumption we conclude that $K[(1,2,\dots,i), z]$ is $(n-i)$-polar. This concludes the proof of (2.a).

(2.b) By Theorem 2.4 of [**8**], for n-polar sets, the properties (2.a) and (2.b) are equivalent. This completes the proof of the theorem.

Using the results of [**3**], it is easy to prove the following proposition.

PROPOSITION 4. *Let v be an n-superharmonic function on Ω, and let δ_i be a regular domain in Ω_i, $i = 1, 2, \dots, n$. For $j = 0, 1, 2, \dots, n$, define v_j on Ω as follows.*

$$v_0(x_1, \dots, x_n) = v(x_1, \dots, x_n),$$

and for $j = 1, 2, \dots, n$, let

$$v_j'(x_1, \dots, x_n) = \begin{cases} \int v_{j-1}(x_1, \dots, x_{j-1}, z, x_{j+1}, \dots, x_n)\, d\rho_{x_j}^{\delta_j}(z) & \text{if } x_j \text{ in } \delta_j, \\ v_{j-1}(x_1, \dots, x_n) & \text{if } x_j \text{ not in } \delta_j. \end{cases}$$

Then,

(1) *v_j is an n-superharmonic function on Ω, and $v_j(x) \leqslant v_{j-1}(x)$ for all x in Ω.*

(2) *v_n is an n-harmonic function on $\delta_1 \times \cdots \times \delta_n$.*

(3) *$v_n(x) = v(x)$ for every x in $(\Omega_1 \setminus \delta_1) \times \cdots \times (\Omega_n \setminus \delta_n)$.*

The following theorem is a generalization, to product of harmonic spaces, of a similar result in a single harmonic space (see Proposition 2.2.3 of [**2**]).

THEOREM 5. *If f is a nonnegative continuous function on Ω with compact support, then MRf is a continuous function on Ω.*

Recall that the support of a real-valued function is the smallest closed set outside which the function is identically zero.

To prove this theorem we need several lemmas.

LEMMA 6. *Let I be an indexing set, and $\{g_i\colon i \text{ in } I\}$ a family of continuous functions on Ω with compact supports. Let there be an $\varepsilon > 0$ such that $|g_i(x) - g_j(x)| \leqslant \varepsilon$ for all x in Ω, and for all i and j in I. Further, let us suppose that one of the following two holds:*

(1) *The constant function* 1 *is n-superharmonic on Ω.*

(2) *There is a relatively compact open set U of Ω such that support of g_i is contained in U for every i in I.*

Then, there is a constant $c > 0$ such that $|MRg_i(x) - MRg_j(x)| < c\varepsilon$ for x in Ω, and for i and j in I. In case (1) *holds, c can be chosen to be* 1, *and in the case where* (2) *holds, c depends only upon U (not on the functions g_i).*

PROOF. If condition (1) holds the proof is trivial. Suppose that (2) holds. Choose a nonnegative continuous function f on Ω with compact support and $f = 1$ on U. Then, MRf is a bounded n-potential on Ω, and hence there is a $c > 0$ such that

$MRf \leqslant c$ on Ω. Observing the fact that $g_i \leqslant g_j + \varepsilon f$ on Ω for i and j, the result follows, and the proof is complete.

LEMMA 7. *Let f be a continuous function on Ω with compact support. Let $1 \leqslant j < n$ be fixed. Further, let d be a metric on $\Omega_1 \times \cdots \times \Omega_j$. For x' in $\Omega_1 \times \cdots \times \Omega_j$, define $f_{x'}$ on $\Omega_{j+1} \times \cdots \times \Omega_n$ as $x'' \mapsto f(x', x'')$. Then given $\varepsilon > 0$, there is an $\eta > 0$ such that $|MRf_{x'}(z) - MRf_{y'}(z)| < \varepsilon$, for every z in $\Omega_{j+1} \times \cdots \times \Omega_n$ and for all x', y' in $\Omega_1 \times \cdots \times \Omega_j$, with $d(x', y') < \eta$. (Note that the multireduced functions $MRf_{x'}$ and $MRf_{y'}$ are defined with respect to the space $\Omega_{j+1} \times \cdots \times \Omega_n$.)*

PROOF OF THEOREM 5. The proof is by induction on n. For $n = 1$, the result is true (see Proposition 2.2.3 of [**2**]). Let us prove the result for the case $n = 2$. Set $v = MRf$. Then, it is obvious that v is lower semicontinuous on Ω. Therefore, it suffices to prove the upper semicontinuity of v by proving that for any (z_1, z_2) in $\Omega_1 \times \Omega_2$

$$\limsup_{\substack{(x,x_2)\to(z_1,z_2)\\ (x_1,x_2) \text{ in } \Omega_1\times\Omega_2}} v(x_1, x_2) \leqslant v(z_1, z_2).$$

Let (z_1, z_2) in $\Omega_1 \times \Omega_2$ be fixed, and let $\varepsilon > 0$. Since f is continuous, and v is lower semicontinuous, there are relatively compact open neighborhoods U_1 and U_2 of (z_1, z_2) with $\overline{U}_1 \subset U_2$, and there is a 2-harmonic function h on $\Omega_1 \times \Omega_2$ satisfying the following conditions.

1. $h(z_1, z_2) = 1$.
2. For all (x_1, x_2) in $\overline{U}_1 \subset U_2$, we have

$$h(x_1, x_2) \geqslant 1 - \varepsilon, \tag{1}$$

$$f(x_1, x_2) \leqslant (f(z_1, z_2) + \varepsilon) h(x_1, x_2), \tag{2}$$

and

$$v(x_1, x_2) \geqslant (v(z_1, z_2) - \varepsilon) h(x_1, x_2). \tag{3}$$

Let d_i be a metric in Ω_i $(i = 1, 2)$. Then, by Lemma 7, there is an $\eta > 0$ such that, $Rf_x(s) - \varepsilon < Rf_y(s) < Rf_x(s) + \varepsilon$ for all s in Ω_1, and

$$Rf_x(t) - \varepsilon < Rf_y(t) < Rf_x(t) + \varepsilon \quad \text{for all } t \text{ in } \Omega_2, \quad \text{if } d_1(x_1, y_1) < \eta, \text{ and } d_2(x_2, y_2) < \eta. \tag{4}$$

Now, choose δ_i, a regular domain in Ω_i, $i = 1, 2$, such that (z_1, z_2) is in $\delta_1 \times \delta_2 \subset \overline{\delta}_1 \times \overline{\delta}_2 \subset U_2$. We may assume that the diameter of δ_i is small enough and it satisfies

$$\int d\rho_{x_i}^{\delta_i}(t) \geqslant 1 - \varepsilon \quad \text{for every } x_i \text{ in } \delta_i,\ i = 1, 2. \tag{5}$$

Put $w = v + 2h$. Then, w is a 2-superharmonic function on $\Omega_1 \times \Omega_2$. Define a function u as follows.

$$u(x_1, x_2) = \begin{cases} \iint w(y_1, y_2)\, d\rho_{x_1}^{\delta_1}(y_1)\, d\rho_{x_2}^{\delta_2}(y_2) & \text{if } (x_1, x_2) \text{ in } \delta_1 \times \delta_2, \\ \int w(x_1, y_2)\, d\rho_{x_2}^{\delta_2}(y_2) & \text{if } x_1 \text{ not in } \delta_1, \text{ and } x_2 \text{ in } \delta_2, \\ \int w(y_1, x_2)\, d\rho_{x_1}^{\delta_1}(y_1) & \text{if } x_1 \text{ in } \delta_1, \text{ and } x_2 \text{ not in } \delta_2, \\ w(x_1, x_2) & \text{otherwise.} \end{cases}$$

It is clear that u is a 2-superharmonic function on $\Omega_1 \times \Omega_2$, $u \leqslant w$, and u is 2-harmonic on $\delta_1 \times \delta_2$.

We claim that $u \geqslant (1 - \varepsilon)f$ on $\Omega_1 \times \Omega_2$. The proof of this claim is given by splitting into four cases, according to whether x_1 is in δ_1 or not, and x_2 is in δ_2 or not.

Case (i). Let (x_1, x_2) be in $\delta_1 \times \delta_2$. Then from the definition of u and w, we have

$$\begin{aligned} u(x_1, x_2) &= \iint \big(v(y_1, y_2) + 2h(y_1, y_2)\big)\, d\rho_{x_1}^{\delta_1}(y_1)\, d\rho_{x_2}^{\delta_2}(y_2) \\ &\geqslant \iint \big(v(z_1, z_2) + \varepsilon\big) h(y_1, y_2)\, d\rho_{x_1}^{\delta_1}(y_1)\, d\rho_{x_2}^{\delta_2}(y_2) \\ &\qquad\qquad \big(\text{using (3) as } \bar{\delta}_1 \times \bar{\delta}_2 \subset U\big) \\ &= \big(v(z_1, z_2) + \varepsilon\big) h(x_1, x_2) \quad (\text{as } h \text{ is 2-harmonic}) \\ &\geqslant \big(f(z_1, z_2) + \varepsilon\big) h(x_1, x_2) \\ &> f(x_1, x_2) \quad (\text{using (2)}) \\ &> (1 - \varepsilon) f(x_1, x_2) \quad (\text{as } f > 0), \end{aligned}$$

Case (ii). Let x_1 be in δ_1 and x_2 not in δ_2. Then,

$$\begin{aligned} u(x_1, x_2) &= \int w(y_1, x_2)\, d\rho_{x_1}^{\delta_1}(y_1) \\ &= \int \big(v(y_1, x_2) + 2\varepsilon h(y_1, x_2)\big)\, d\rho_{x_1}^{\delta_1}(y_1) \\ &\geqslant \int \big(Rf_{y_1}(x_2) + 2\varepsilon h(y_1, x_2)\big)\, d\rho_{x_1}^{\delta_1}(y_1) \\ &\geqslant \int \big(Rf_{x_1}(x_2) - \varepsilon + 2\varepsilon h(y_1, x_2)\big)\, d\rho_{x_1}^{\delta_1}(y_1) \quad (\text{using (4)}) \\ &= \big(Rf_{x_1}(x_2) - \varepsilon\big) \int d\rho_{x_1}^{\delta_1}(y_1) + 2\varepsilon h(x_1, x_2) \\ &> \big(Rf_{x_1}(x_2) - \varepsilon\big)(1 - \varepsilon) + 2\varepsilon h(x_1, x_2) \quad (\text{using (5)}) \\ &> \big(f_{x_1}(x_2) - \varepsilon\big)(1 - \varepsilon) + 2\varepsilon h(x_1, x_2) \quad \big(\text{as } Rf_{x_1} > f_{x_1} \text{ on } \Omega_2\big) \\ &= \big(f(x_1, x_2) - \varepsilon\big)(1 - \varepsilon) + 2\varepsilon(1 - \varepsilon) \quad (\text{using (1)}) \\ &> (1 - \varepsilon) f(x_1, x_2) \quad (\text{as } f > 0). \end{aligned}$$

Case (iii). Let x_1 not be in δ_1 and x_2 in δ_2. The proof is similar to the previous case.

Case (iv). Let x_1 not be in δ_1 and x_2 not in δ_2. Then the proof trivially follows from the definitions of u, v and w. Thus, the claim is proved.

Now, $u \geqslant (1-\varepsilon)f$ on $\Omega_1 \times \Omega_2$ gives that $u \geqslant (1-\varepsilon)v$ on $\Omega_1 \times \Omega_2$. In particular, if (x_1, x_2) is in $\delta_1 \times \delta_2$ then

$$(1-\varepsilon)v(x_1, x_2) \leqslant \iint v(y_1, y_2)\, d\rho_{x_1}^{\delta_1}(y_1)\, d\rho_{x_2}^{\delta_2}(y_2) + 2h(x_1, x_2).$$

The right-hand side of the above inequality is a 2-harmonic function on $\delta_1 \times \delta_2$, hence is a continuous function on $\delta_1 \times \delta_2$. Taking lim sup as $(x_1, x_2) \to (z_1, z_2)$ in $\Omega_1 \times \Omega_2$, and noticing that v is a 2-superharmonic function we get

$$\begin{aligned}(1-\varepsilon) \limsup_{\substack{(x_1, x_2) \to (z_1, z_2) \\ (x_1, x_1) \text{ in } \Omega_1 \times \Omega_2}} v(x_1, x_2) &\leqslant v(z_1, z_2) + 2\varepsilon h(z_1, z_2) \\ &\leqslant v(z_1, z_2) + 2\varepsilon.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{(x_1, x_2) \to (z_1, z_2)} v(x_1, x_2) \leqslant v(z_1, z_2).$$

Since (z_1, z_2) is an arbitrary point in $\Omega_1 \times \Omega_2$, it follows that v is upper semicontinuous on $\Omega_1 \times \Omega_2$. Hence, v is continuous on $\Omega_1 \times \Omega_2$, and this concludes the proof for the case $n = 2$.

To complete the proof of the induction, we proceed from the case of functions of $n-1$ variables to functions of n variables in exactly the same way. We remark that the choice of u in the above proof is replaced by w_n as defined in Proposition 4. The rest of the details are absolutely the same. This allows us to conclude that MRf is in general a continuous function whenever f is a nonnegative continuous function with compact support, completing the proof of the theorem.

As an immediate consequence, we have the following corollary.

COROLLARY 8. *If f is a nonnegative continuous function on Ω with compact support, then MRf is a continuous n-potential on Ω.*

Though the following result is essentially a corollary to the above theorem, we will state it as a theorem due to its importance. We omit the proof.

THEOREM 9. *Let v be a positive n-superharmonic function on Ω. Then, there is a sequence v_j of continuous n-potentials such that v_j increases pointwise to v on Ω as $j \to \infty$.*

From now on K is a compact n-polar subset of Ω, such that every i-section of K through any point of Ω is $(n-i)$-polar, for $i = 1, 2, \ldots, n-1$.

The next theorem is the converse of Theorem 3.

THEOREM 10. *Given a positive continuous function f on K and an $\varepsilon > 0$, there exists a continuous n-potential p on Ω such that $|f - p| < \varepsilon$ on K.*

PROOF. Let $\{U_j\}$, $j = 1, 2, 3, \ldots$, be a decreasing sequence of relatively compact subsets of Ω such that $U_j \supset \overline{U}_{j+1} \supset U_{j+1}$ for $j = 1, 2, \ldots$, and $K = \bigcap\{U_j\colon j = 1, 2, 3, \ldots\}$. For each j, let f_j be a nonnegative continuous extension of f to Ω with support of $f_j \subset U_j$. By taking infimum at each stage, we may assume that $\{f_j\}$ is a decreasing sequence of functions on Ω.

Put $p_j = MRf_j$. Then, by Corollary 8, p_j is a continuous n-potential, for each j. Following the proof Theorem 4 of [7], we can show that p_j decreases pointwise to f on k as $j \to \infty$. Using Dini's Theorem, we conclude that p_j converges to f uniformly on K. Thus, there is an m such that $|f(x) - p_j(x)| < \varepsilon$ if $j \geqslant m$, for all x in K. The choice $p = p_m$ meets the requirement of the theorem, completing the proof.

The following result is an analogue of Theorem 5 of [7], and can be proved analogously with the help of Theorem 10.

PROPOSITION 11. *Let f_0 be a positive continuous function on K, and F_0 be a relatively compact open neighborhood of K. Put $F = \overline{F}_0$, and let f be a nonnegative continuous extension of f_0 to Ω, such that $f > 0$ on F. Then, given $\varepsilon > 0$, there exists a continuous potential p on Ω such that $p < f$ on F, and $p \geqslant f_0 - \varepsilon$ on K.*

Our ultimate aim is the following theorem, for the case $n \geqslant 2$.

THEOREM 12. *Given a positive continuous function f_0 on K, there is a continuous n-potential p on Ω such that $p = f_0$ on K.*

PROOF. The existence of an n-potential p such that $p = f_0$ on K can be proved as in the case $n = 1$. (See Theorem 2 of [7].) However, in proving the continuity of p, in the case $n = 1$, we have explicitly used the fact that Rg is harmonic outside the support of g. This result is no longer valid for MRg when $n > 1$. Hence, we modify the proof as follows. We also note that the same method works in the case $n = 1$.

Let $\varepsilon > 0$. For a continuous function g on K, define

$$\|g\|_K = \sup\{|g(x)|\colon x \text{ in } K\},$$

and if g is a bounded continuous function on Ω, then define

$$\|g\|_\infty = \sup\{|g(x)|\colon x \text{ in } \Omega\}.$$

Let q be a bounded continuous n-potential on Ω. We may assume that $q \geqslant 1$ on K.

Let F_0 be a relatively compact open set containing K and let $F = \overline{F}_0$. Choose f a nonnegative continuous extension of f_0 to Ω with $f > 0$ on F, and $\|f\|_\infty = \|f_0\|_K$. Then, by the previous theorem, there is a continuous n-potential q_0 on Ω such that $q_0 < f$ on F and $q_0 > f_0 - \varepsilon$ on K. Let $p_0 = \inf\{\|f_0\|_K q, q_0\}$ on Ω. Then, p_0 is a bounded continuous n-potential on Ω. Further, $p_0 \leqslant q_0 < f$ on F. If x is in K, then $q(x) \geqslant 1$, and hence,

$$\|f_0\|_K q(x) \geqslant \|f_0\|_K = \|f\|_\infty \geqslant f(x) > q_0(x).$$

Therefore, $p_0 = q_0$ on K and hence, $p_0 \geqslant f_0 - \varepsilon$ on K. Thus, there is a bounded continuous n-potential p_0 such that

1. $p_0(x) < f(x)$ for all x in F,
2. $p_0(x) \geqslant f_0(x) - \varepsilon$ for all x in K,
3. $\|p_0\|_\infty < \|f_0\|_K \|q\|_\infty$.

Put $g_1 = \max(f - p_0, 0)$ on Ω. Then, g_1 is a nonnegative continuous function and $g_1 > 0$ on F. Let g_2 be a nonnegative continuous on Ω such that $g_2 = g_1$ on K and $g_2 > 0$ on F. Since $g_2 = g_1 = f - p_0$ on K, we may even choose g_2 such that $\|g_2\|_\infty = \|f - p_0\|_K$. Set $f_1 = \inf\{g_1, g_2\}$. Then, f_1 is a nonnegative continuous function on Ω with $f_1 > 0$ on F and $f_1 = f - p_0$ on K. As before, there is a continuous n-potential p_1 such that

1. $p_1(x) < f_1(x)$ for every x in F,
2. $p_1(x) \geqslant f_1(x) - \varepsilon/2$ for every x in K,
3. $\|p_1\|_\infty \leqslant \|f_1\|_K \|q\|_\infty$.

Now, $f_1(x) \leqslant g_1(x) \leqslant f(x) - p_0(x)$ on F, and $f_1(x) = g_1(x) = f(x) - p_0(x)$ on K.

Therefore, the above inequalities can be rewritten as follows.

1. $p_0(x) + p_1(x) < f(x)$ for all x in F,
2. $p_0(x) + p_1(x) \geqslant f_0(x) - \varepsilon/2$ for all x in K,
3. $\|p_0\|_\infty \leqslant \|f_0\|_K \|q\|_\infty$ and $\|p_1\|_\infty \leqslant \|f_1\|_K \|q\|_\infty$.

Note that $\|f_1\|_K \leqslant \varepsilon/2$.

Proceeding by induction, we get the sequence $\{p_m\}$, $m = 0, 1, 2, \ldots,$ of bounded continuous n-potentials and a sequence $\{f_m\}$, $m = 0, 1, 2, \ldots,$ of continuous functions such that

1. $\sum_{i=0}^m p_i < f$ on F for every m,
2. $\sum_{i=0}^m p_i > f_0 - \varepsilon/2^m$ on K for every m,
3. $\|p_m\|_\infty \leqslant \|f_m\|_K \|q\|_\infty$ for $m = 0, 1, 2, \ldots$.

Note that $\|f_m\|_K < \varepsilon/2^m$ for $m = 1, 2, 3, \ldots$.

Set $p = \sum_{i=0}^\infty p_m$ on Ω. Then, it is clear that p is an n-superharmonic function and that $p \leqslant f$ on F and $p = f_0$ on K. By an analogue of Proposition 2.2.2 of **[2]**, p is an n-potential on Ω. As $\|p_m\|_\infty \leqslant \|f_m\|_K \|q\|_\infty \leqslant (\varepsilon/2^m)\|q\|_\infty$ for $m \geqslant 1$, $\sum_{m=0}^\infty p_m(x)$ converges uniformly on Ω. Since each p_m is a continuous function, p is continuous on Ω, completing the proof.

The following corollary is an immediate consequence of the above theorem.

COROLLARY 13. (1) *Every real-valued continuous function on K is the restriction to K of the difference of two positive continuous n-potentials on* Ω.

(2) *Every positive lower semicontinuous function on K is the restriction to K of an n-potential.*

(3) *If the constant function* 1 *is n-superharmonic on* Ω, *then every real-valued continuous function on K is the restriction to K of an n-superharmonic function on* Ω.

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