CONTINUOUS FUNCTIONS ON MULTIPOLAR SETS

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Abstract. Let \( \Omega = \Omega_1 \times \cdots \times \Omega_n \) (\( n > 1 \)) be a product of \( n \) Brelot harmonic spaces each of which has a bounded potential, and let \( K \) be a compact subset of \( \Omega \). Then, \( K \) is an \( n \)-polar set with the property that every \( i \)-section (\( 1 \leq i < n \)) of \( K \) through any point in \( \Omega \) is \((n - i)\)-polar if and only if every positive continuous function on \( K \) can be extended to a continuous potential on \( \Omega \). Further, it has been shown that if \( f \) is a nonnegative continuous function on \( \Omega \) with compact support, then \( MRf \), the multireduced function of \( f \) over \( \Omega \), is also a continuous function on \( \Omega \).

1. Introduction. Let \( \Omega_j \) for \( j = 1, 2, \ldots, n \) be locally compact spaces with countable basis for the topology and be Brelot spaces [5]. The principal results of this paper (cf. Theorem 3 and Theorem 12) characterize certain exceptional compact sets \( K \) contained in the product space \( \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \) in terms of extendibility of positive continuous functions on \( K \), to a positive \( n \)-potential on the entire space. This is a natural generalization of the corresponding results in [7] in the case of a single harmonic space. A key result, independent of interest, needed in the proof of the main results in Theorem 5 shows that the multireduced function of a continuous function is continuous. The results of this paper formed a part of my Ph. D. thesis submitted to McGill University [6]. I would like to thank Professor Gowrisankaran for his help in preparation of this work.

We shall use the notation and results from [3, 4 and 8] concerning \( n \)-harmonic, \( n \)-superharmonic functions, and \( n \)-potentials. We assume that there is a bounded \( n \)-superharmonic function on \( \Omega_1 \times \cdots \times \Omega_n \). Notice that it is equivalent to the assumption that each \( \Omega_j \) (\( j = 1, \ldots, n \)) has a bounded potential. If constants are superharmonic, then this assumption trivially holds. Throughout this paper, unless it is explicitly mentioned, \( n \) is an integer \( \geq 2 \), and we denote \( \Omega_1 \times \cdots \times \Omega_n \) by \( \Omega \).

In the course of proving our main results we need a number of results that are routine generalizations of similar ones in the single variable case. The proofs of some of them are not quite obvious, and can be found in [6].

Definition 1. Let \( f \) be an extended real-valued function on \( \Omega \). Define \( MRf(x) = \inf \{ u(x) : u \text{ is } n\text{-hyperharmonic, and } u \geq f \text{ on } \Omega \} \). For each subset \( E \) of \( \Omega \), let \( X_E \) be the characteristic function of \( E \), and let \( X_E \cdot f \) be the pointwise product function on \( \Omega \). The function \( MR(X_E \cdot f) \) is called the multireduced function of \( f \) over \( E \).
Observe that \((\text{MR}(X_E \cdot f))^{\dagger}\), the lower semicontinuous regularization (see [1]) of \(\text{MR}(X_E \cdot f)\), is an \(n\)-hyperharmonic function on \(\Omega\).

Let us recall the definition of a section from [8].

**Definition 2.** Let \(E\) be a subset of \(\Omega\), \(n > 1\), and \(i\) an integer such that \(1 \leq i \leq n - 1\). Let \(r_1, \ldots, r_i\) be integers with \(1 \leq r_1 < r_2 < \cdots < r_i \leq n\). Let \(t = (x_{s_1}, \ldots, x_{s_i})\) be a point in \(\Omega_{s_1} \times \cdots \times \Omega_{s_i}\) \((1 \leq j \leq n)\) with \(\{r_1, \ldots, r_i\} \subseteq \{s_1, \ldots, s_j\} \subseteq \{1, 2, \ldots, n\}\). Then the \((r_1, \ldots, r_i)\)-section of \(E\) through \(t\) is defined as

\[
z \in \prod_{k=1}^{n} \Omega_k, \quad (y_1, \ldots, y_n) \text{ is in } E, \quad \text{whenever } y_k = x_k \text{ for } k \in \{r_1, \ldots, r_i\},
\]

\[
k \text{ not in } \{r_1, \ldots, r_i\}, \quad y_k = z_k \text{ for } k \text{ not in } \{r_1, \ldots, r_i\},
\]

which would be denoted by \(E[(r_1, \ldots, r_i), t]\). For \(x\) in \(\Omega\), an \(i\)-section of \(E\) through \(x\) is always denoted by \(E[(r_1, \ldots, r_i), x]\) for some \((r_1, \ldots, r_i)\).

2. **A characterization of a class of multipolar sets.** The following theorem gives a necessary condition for a compact set to be \(n\)-polar. We recall that a set \(E\) is \(n\)-polar if there exists a positive \(n\)-superharmonic function which is identically infinity on \(E\).

**Theorem 3.** Let \(K\) be a compact subset of \(\Omega\) such that every positive continuous function on \(K\) can be uniformly approximated on \(K\) by positive \(n\)-superharmonic functions on \(\Omega\). Then,

(1) If \(K\) has more than one point, then \(K\) is an \(n\)-polar set.

(2) In addition, if point sets are polar in \(\Omega_j\) for \(j = 1, 2, \ldots, n\), then

(a) Every \(i\)-section of \(K\) through any point in \(\Omega\) is \((n-i)\)-polar for \(i = 1, 2, \ldots, n - 1\).

(b) Given \(z\) in \(\Omega\) such that \(z\) not in \(K\), there is a positive \(n\)-superharmonic function \(u\) on \(\Omega\) such that \(u(z) < \infty\) and \(u(x) = \infty\) for all \(x\) in \(K\).

**Proof.** (1) The proof of the fact that the hypothesis implies the \(n\)-polarity of \(K\) is very similar to the proof of Theorem 1 of [7].

(2.a) This part is proved by induction on \(n\). Let us make the induction assumption that for all products of Brelot spaces \(\Omega_1 \times \cdots \times \Omega_m\) with \(m < n\), the hypothesis of the theorem implies the property (a).

Let us now consider a compact set \(K\) contained in \(\Omega\) satisfying the hypothesis. Fix \(z\) in \(\Omega\), and \(i\) such that \(1 \leq i \leq n - 1\). By a suitable rearrangement, if necessary, we may consider the \(i\)-section \(K[(1, 2, \ldots, i), z]\) to be the general \(i\)-section of \(K\). This set is compact and contained in \(\Omega_{i+1} \times \cdots \times \Omega_n\). Let us consider further the nontrivial case when \(K[(1, 2, \ldots, i), z]\) contains more than one point. We shall show that this set is \((n-i)\)-polar. By the induction hypothesis, it suffices to prove that an arbitrary positive continuous function \(f\) on \(K[(1, 2, \ldots, i), z]\) can be uniformly approximated on \(K[(1, 2, \ldots, i), z]\) by positive \((n-i)\)-superharmonic functions.

Let us consider the copy \(L[i, z]\) of the above compact set in \(K\), viz., \(L[i, z] = \{(z_1, \ldots, z_i, y_{i+1}, \ldots, y_n) \text{ in } K\}\). \(L[i, z]\) is a compact subset of \(K\), and \(f\) can be considered as a positive continuous function on this set. By the Tietze extension theorem, there exists a positive continuous function \(g\) on \(K\) which extends \(f\). By
hypothesis, for \( \varepsilon > 0 \), there exists a positive \( n \)-superharmonic function \( u \) on \( \Omega \) such that \( |u - g| < \varepsilon \) on \( K \) uniformly. Set

\[
v(y_{i+1}, \ldots, y_n) = u(z_1, \ldots, z_i, y_{i+1}, \ldots, y_n).
\]

This function is \( (n - i) \)-superharmonic on \( \Omega_{i+1} \times \cdots \times \Omega_n \), and clearly \( |f - v| < \varepsilon \) uniformly on \( K[(1, 2, \ldots, i), z] \). Hence, by the induction assumption we conclude that \( K[(1, 2, \ldots, i), z] \) is \( (n - i) \)-polar. This concludes the proof of (2.a).

(2.b) By Theorem 2.4 of [8], for \( n \)-polar sets, the properties (2.a) and (2.b) are equivalent. This completes the proof of the theorem.

Using the results of [3], it is easy to prove the following proposition.

**Proposition 4.** Let \( v \) be an \( n \)-superharmonic function on \( \Omega \), and let \( \delta_i \) be a regular domain in \( \Omega_i \), \( i = 1, 2, \ldots, n \). For \( j = 0, 1, 2, \ldots, n \), define \( v_j \) on \( \Omega \) as follows.

\[
v_0(x_1, \ldots, x_n) = v(x_1, \ldots, x_n),
\]

and for \( j = 1, 2, \ldots, n \), let

\[
v_j(x_1, \ldots, x_n) = \begin{cases} 
\int v_{j-1}(x_1, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_n) \, dp_{\delta_j}(z) & \text{if } x_j \text{ in } \delta_j, \\
v_{j-1}(x_1, \ldots, x_n) & \text{if } x_j \text{ not in } \delta_j.
\end{cases}
\]

Then,

1. \( v_j \) is an \( n \)-superharmonic function on \( \Omega \), and \( v_j(x) \leq v_{j-1}(x) \) for all \( x \) in \( \Omega \).
2. \( v_n \) is an \( n \)-harmonic function on \( \delta_1 \times \cdots \times \delta_n \).
3. \( v_n(x) = v(x) \) for every \( x \) in \( (\Omega_1 \setminus \delta_1) \times \cdots \times (\Omega_n \setminus \delta_n) \).

The following theorem is a generalization, to product of harmonic spaces, of a similar result in a single harmonic space (see Proposition 2.2.3 of [2]).

**Theorem 5.** If \( f \) is a nonnegative continuous function on \( \Omega \) with compact support, then \( MRf \) is a continuous function on \( \Omega \).

Recall that the support of a real-valued function is the smallest closed set outside which the function is identically zero.

To prove this theorem we need several lemmas.

**Lemma 6.** Let \( I \) be an indexing set, and \( \{ g_i : i \in I \} \) a family of continuous functions on \( \Omega \) with compact supports. Let there be an \( \varepsilon > 0 \) such that \( |g_i(x) - g_j(x)| \leq \varepsilon \) for all \( x \) in \( \Omega \), and for all \( i \) and \( j \) in \( I \). Further, let us suppose that one of the following two holds:

1. The constant function 1 is \( n \)-superharmonic on \( \Omega \).
2. There is a relatively compact open set \( U \) of \( \Omega \) such that support of \( g_i \) is contained in \( U \) for every \( i \) in \( I \).

Then, there is a constant \( c > 0 \) such that \( |MRg_i(x) - MRg_j(x)| \leq c\varepsilon \) for \( x \) in \( \Omega \), and for \( i \) and \( j \) in \( I \). In case (1) holds, \( c \) can be chosen to be 1, and in the case where (2) holds, \( c \) depends only upon \( U \) (not on the functions \( g_i \)).

**Proof.** If condition (1) holds the proof is trivial. Suppose that (2) holds. Choose a nonnegative continuous function \( f \) on \( \Omega \) with compact support and \( f = 1 \) on \( U \). Then, \( MRf \) is a bounded \( n \)-potential on \( \Omega \), and hence there is a \( c > 0 \) such that
MRf ≤ c on Ω. Observing the fact that $g_i ≤ g_j + ef$ on Ω for i and j, the result follows, and the proof is complete.

**Lemma 7.** Let f be a continuous function on Ω with compact support. Let 1 < j < n be fixed. Further, let d be a metric on $Ω_1 × \cdots × Ω_j$. For $x'$ in $Ω_1 × \cdots × Ω_j$, define $f_{x'}$ on $Ω_{j+1} × \cdots × Ω_n$ as $x'' → f(x', x'')$. Then given $ε > 0$, there is an $η > 0$ such that $|MRf_{x'}(z) - MRf_{y'}(z)| < ε$, for every z in $Ω_{j+1} × \cdots × Ω_n$ and for all $x', y'$ in $Ω_1 × \cdots × Ω_j$, with $d(x', y') < η$. (Note that the multireduced functions $MRf_{x'}$ and $MRf_{y'}$ are defined with respect to the space $Ω_{j+1} × \cdots × Ω_n$.)

**Proof of Theorem 5.** The proof is by induction on n. For n = 1, the result is true (see Proposition 2.2.3 of [2]). Let us prove the result for the case n = 2. Set $v = MRf$. Then, it is obvious that v is lower semicontinuous on Ω. Therefore, it suffices to prove the upper semicontinuity of v by proving that for any $(z_1, z_2)$ in $Ω_1 × Ω_2$

$$\limsup_{(x, x_2) \to (z_1, z_2)} v(x_1, x_2) \leq v(z_1, z_2).$$

Let $(z_1, z_2)$ in $Ω_1 × Ω_2$ be fixed, and let $ε > 0$. Since f is continuous, and v is lower semicontinuous, there are relatively compact open neighborhoods $U_1$ and $U_2$ of $(z_1, z_2)$ with $U_1 \subset U_2$, and there is a 2-harmonic function $h$ on $Ω_1 × Ω_2$ satisfying the following conditions.

1. $h(z_1, z_2) = 1$.
2. For all $(x_1, x_2)$ in $U_1 \subset U_2$, we have

$$f(x_1, x_2) \leq (f(z_1, z_2) + ε)h(x_1, x_2),$$

and

$$v(x_1, x_2) ≥ (v(z_1, z_2) - ε)h(x_1, x_2).$$

Let $d_i$ be a metric in $Ω_i$, (i = 1, 2). Then, by Lemma 7, there is an $η > 0$ such that, $Rf_x(s) - ε < Rf_x(s) < Rf_x(s) + ε$ for all s in $Ω_1$, and

$$Rf_x(t) - ε < Rf_x(t) < Rf_x(t) + ε \quad \text{for all } t \text{ in } Ω_2,$$

if $d_1(x_1, y_1) < η$, and $d_2(x_2, y_2) < η$.

Now, choose $δ_i$, a regular domain in $Ω_i$, i = 1, 2, such that $(z_1, z_2)$ is in $δ_1 × δ_2 \subset δ_1 × δ_2 \subset U_2$. We may assume that the diameter of $δ_i$ is small enough and it satisfies

$$\int dρ_{δ_i}^{\delta_i}(t) ≥ 1 - ε \quad \text{for every } x_i \text{ in } δ_i, \ i = 1, 2.$$
Put \( w = v + 2h \). Then, \( w \) is a 2-superharmonic function on \( \Omega_1 \times \Omega_2 \). Define a function \( u \) as follows.

\[
\begin{align*}
  u(x_1, x_2) &= \begin{cases} \\
  \int\int w(y_1, y_2) \, d\rho^\delta_{x_1}(y_1) \, d\rho^\delta_{x_2}(y_2) & \text{if } (x_1, x_2) \text{ in } \delta_1 \times \delta_2, \\
  \int w(x_1, y_2) \, d\rho^\delta_{x_2}(y_2) & \text{if } x_1 \text{ not in } \delta_1, \text{ and } x_2 \text{ in } \delta_2, \\
  \int w(y_1, x_2) \, d\rho^\delta_{x_1}(y_1) & \text{if } x_1 \text{ in } \delta_1, \text{ and } x_2 \text{ not in } \delta_2, \\
  w(x_1, x_2) & \text{otherwise.}
  \end{cases}
\end{align*}
\]

It is clear that \( u \) is a 2-superharmonic function on \( \Omega_1 \times \Omega_2 \), \( u \leq w \), and \( u \) is 2-harmonic on \( \delta_1 \times \delta_2 \).

We claim that \( u \geq (1 - \epsilon)f \) on \( \Omega_1 \times \Omega_2 \). The proof of this claim is given by splitting into four cases, according to whether \( x_1 \) is in \( \delta_1 \) or not, and \( x_2 \) is in \( \delta_2 \) or not.

**Case (i).** Let \((x_1, x_2)\) be in \(\delta_1 \times \delta_2\). Then from the definition of \(u\) and \(w\), we have

\[
  u(x_1, x_2) = \int\int (v(y_1, y_2) + 2h(y_1, y_2)) \, d\rho^\delta_{x_1}(y_1) \, d\rho^\delta_{x_2}(y_2)
  \geq \int\int (v(z_1, z_2) + \epsilon)h(y_1, y_2) \, d\rho^\delta_{x_1}(y_1) \, d\rho^\delta_{x_2}(y_2)
  = (v(z_1, z_2) + \epsilon)h(x_1, x_2) \quad (\text{as } h \text{ is 2-harmonic})
  \geq (f(z_1, z_2) + \epsilon)h(x_1, x_2)
  > f(x_1, x_2) \quad (\text{using (2)})
  > (1 - \epsilon)f(x_1, x_2) \quad (\text{as } f > 0),
\]

**Case (ii).** Let \( x_1 \) be in \( \delta_1 \) and \( x_2 \) not in \( \delta_2 \). Then,

\[
  u(x_1, x_2) = \int w(y_1, x_2) \, d\rho^\delta_{x_1}(y_1)
  = \int (v(y_1, x_2) + 2\epsilon h(y_1, x_2)) \, d\rho^\delta_{x_1}(y_1)
  \geq \int (Rf_{x_1}(x_2) + 2\epsilon h(y_1, x_2)) \, d\rho^\delta_{x_1}(y_1)
  \geq \int (Rf_{x_1}(x_2) - \epsilon + 2\epsilon h(y_1, x_2)) \, d\rho^\delta_{x_1}(y_1) \quad (\text{using (4)})
  = (Rf_{x_1}(x_2) - \epsilon) \int d\rho^\delta_{x_1}(y_1) + 2\epsilon h(x_1, x_2)
  \geq (Rf_{x_1}(x_2) - \epsilon)(1 - \epsilon) + 2\epsilon h(x_1, x_2) \quad (\text{using (5)})
  > (f(x_1, x_2) - \epsilon)(1 - \epsilon) + 2\epsilon h(x_1, x_2) \quad (\text{as } Rf_{x_1} > f_{x_1} \text{ on } \Omega_2)
  \geq (f(x_1, x_2) - \epsilon)(1 - \epsilon) + 2\epsilon(1 - \epsilon) \quad (\text{using (1)})
  > (1 - \epsilon)f(x_1, x_2) \quad (\text{as } f > 0).
\]
Case (iii). Let $x_1$ not be in $\delta_1$ and $x_2$ in $\delta_2$. The proof is similar to the previous case.

Case (iv). Let $x_1$ not be in $\delta_1$ and $x_2$ not in $\delta_2$. Then the proof trivially follows from the definitions of $u, v$ and $w$. Thus, the claim is proved.

Now, $u \geq (1 - \epsilon)f$ on $\Omega_1 \times \Omega_2$ gives that $u \geq (1 - \epsilon)v$ on $\Omega_1 \times \Omega_2$. In particular, if $(x_1, x_2)$ is in $\delta_1 \times \delta_2$ then

$$(1 - \epsilon)u(x_1, x_2) \leq \int \int v(y_1, y_2) \, d\rho^{\delta_1}_{x_1}(y_1) \, d\rho^{\delta_2}_{x_2}(y_2) + 2h(x_1, x_2).$$

The right-hand side of the above inequality is a 2-harmonic function on $\delta_1 \times \delta_2$, hence is a continuous function on $\delta_1 \times \delta_2$. Taking lim sup as $(x_1, x_2) \rightarrow (z_1, z_2)$ in $\Omega_1 \times \Omega_2$, and noticing that $v$ is a 2-superharmonic function we get

$$(1 - \epsilon) \limsup_{(x_1, x_2) \rightarrow (z_1, z_2)} v(x_1, x_2) \leq v(z_1, z_2) + 2\epsilon h(z_1, z_2)$$

As $\epsilon > 0$ is arbitrary, we have

$$\limsup_{(x_1, x_2) \rightarrow (z_1, z_2)} v(x_1, x_2) \leq v(z_1, z_2).$$

Since $(z_1, z_2)$ is an arbitrary point in $\Omega_1 \times \Omega_2$, it follows that $v$ is upper semicontinuous on $\Omega_1 \times \Omega_2$. Hence, $v$ is continuous on $\Omega_1 \times \Omega_2$, and this concludes the proof for the case $n = 2$.

To complete the proof of the induction, we proceed from the case of functions of $n - 1$ variables to functions of $n$ variables in exactly the same way. We remark that the choice of $u$ in the above proof is replaced by $w_n$ as defined in Proposition 4. The rest of the details are absolutely the same. This allows us to conclude that $MRf$ is in general a continuous function whenever $f$ is a nonnegative continuous function with compact support, completing the proof of the theorem.

As an immediate consequence, we have the following corollary.

**Corollary 8.** If $f$ is a nonnegative continuous function on $\Omega$ with compact support, then $MRf$ is a continuous $n$-potential on $\Omega$.

Though the following result is essentially a corollary to the above theorem, we will state it as a theorem due to its importance. We omit the proof.

**Theorem 9.** Let $v$ be a positive $n$-superharmonic function on $\Omega$. Then, there is a sequence $v_j$ of continuous $n$-potentials such that $v_j$ increases pointwise to $v$ on $\Omega$ as $j \rightarrow \infty$.

From now on $K$ is a compact $n$-polar subset of $\Omega$, such that every $i$-section of $K$ through any point of $\Omega$ is $(n - i)$-polar, for $i = 1, 2, \ldots, n - 1$.

The next theorem is the converse of Theorem 3.

**Theorem 10.** Given a positive continuous function $f$ on $K$ and an $\epsilon > 0$, there exists a continuous $n$-potential $p$ on $\Omega$ such that $|f - p| < \epsilon$ on $K$. 

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Proof. Let \( \{U_j\} \), \( j = 1, 2, 3, \ldots, \) be a decreasing sequence of relatively compact subsets of \( \Omega \) such that \( U_j \supset U_{j+1} \supset U_{j+1} \) for \( j = 1, 2, \ldots, \) and \( K = \cap \{U_j: j = 1, 2, 3, \ldots\} \). For each \( j \), let \( f_j \) be a nonnegative continuous extension of \( f \) to \( \Omega \) with support of \( f_j \subset U_j \). By taking infimum at each stage, we may assume that \( \{f_j\} \) is a decreasing sequence of functions on \( \Omega \).

Put \( p_j = MRf_j \). Then, by Corollary 8, \( p_j \) is a continuous \( n \)-potential, for each \( j \). Following the proof Theorem 4 of [7], we can show that \( p_j \) decreases pointwise to \( f \) on \( k \) as \( j \to \infty \). Using Dini's Theorem, we conclude that \( p_j \) converges to \( f \) uniformly on \( K \). Thus, there is an \( m \) such that \( |f(x) - p_j(x)| < \epsilon \) if \( j \geq m \), for all \( x \) in \( K \). The choice \( p = p_m \) meets the requirement of the theorem, completing the proof.

The following result is an analogue of Theorem 5 of [7], and can be proved analogously with the help of Theorem 10.

Proposition 11. Let \( f_0 \) be a positive continuous function on \( K \), and \( F_0 \) be a relatively compact open neighborhood of \( K \). Put \( F = \mathring{F}_0 \), and let \( f \) be a nonnegative continuous extension of \( f_0 \) to \( \Omega \), such that \( f > 0 \) on \( F \). Then, given \( \epsilon > 0 \), there exists a continuous potential \( p \) on \( \Omega \) such that \( p < f \) on \( F \), and \( p \geq f_0 - \epsilon \) on \( K \).

Our ultimate aim is the following theorem, for the case \( n \geq 2 \).

Theorem 12. Given a positive continuous function \( f_0 \) on \( K \), there is a continuous \( n \)-potential \( p \) on \( \Omega \) such that \( p = f_0 \) on \( K \).

Proof. The existence of an \( n \)-potential \( p \) such that \( p = f_0 \) on \( K \) can be proved as in the case \( n = 1 \). (See Theorem 2 of [7].) However, in proving the continuity of \( p \), in the case \( n = 1 \), we have explicitly used the fact that \( Rg \) is harmonic outside the support of \( g \). This result is no longer valid for \( MRg \) when \( n > 1 \). Hence, we modify the proof as follows. We also note that the same method works in the case \( n = 1 \).

Let \( \epsilon > 0 \). For a continuous function \( g \) on \( K \), define

\[
\|g\|_K = \sup\{|g(x)|: x \text{ in } K\},
\]
and if \( g \) is a bounded continuous function on \( \Omega \), then define

\[
\|g\|_\infty = \sup\{|g(x)|: x \text{ in } \Omega\}.
\]

Let \( q \) be a bounded continuous \( n \)-potential on \( \Omega \). We may assume that \( q \geq 1 \) on \( K \).

Let \( F_0 \) be a relatively compact open set containing \( K \) and let \( F = \mathring{F}_0 \). Choose \( f \) a nonnegative continuous extension of \( f_0 \) to \( \Omega \) with \( f > 0 \) on \( F \), and \( \|f\|_\infty = \|f_0\|_K \).

Then, by the previous theorem, there is a continuous \( n \)-potential \( q_0 \) on \( \Omega \) such that \( q_0 < f \) on \( F \) and \( q_0 > f_0 - \epsilon \) on \( K \). Let \( p_0 = \inf\{\|f_0\|_K q, q_0\} \) on \( \Omega \). Then, \( p_0 \) is a bounded continuous \( n \)-potential on \( \Omega \). Further, \( p_0 \leq q_0 < f \) on \( F \). If \( x \) is in \( K \), then \( q(x) \geq 1 \), and hence,

\[
\|f_0\|_K q(x) \geq \|f_0\|_K = \|f\|_\infty \geq f(x) > q_0(x).
\]
Therefore, \( p_0 = q_0 \) on \( K \) and hence, \( p_0 \geq f_0 - \epsilon \) on \( K \). Thus, there is a bounded continuous \( n \)-potential \( p_0 \) such that
1. \( p_0(x) < f(x) \) for all \( x \) in \( F \),
2. \( p_0(x) \geq f_0(x) - \epsilon \) for all \( x \) in \( K \),
3. \( \|p_0\|_\infty < \|f_0\|_K \|q\|_\infty \).

Put \( g_1 = \max(f - p_0, 0) \) on \( \Omega \). Then, \( g_1 \) is a nonnegative continuous function and \( g_1 > 0 \) on \( F \). Let \( g_2 \) be a nonnegative continuous on \( \Omega \) such that \( g_2 = g_1 \) on \( K \) and \( g_2 > 0 \) on \( F \). Since \( g_2 = g_1 = f - p_0 \) on \( K \), we may even choose \( g_2 \) such that \( \|g_2\|_\infty = \|f - p_0\|_K \). Set \( f_1 = \inf\{g_1, g_2\} \). Then, \( f_1 \) is a nonnegative continuous function on \( \Omega \) with \( f_1 > 0 \) on \( F \) and \( f_1 = f - p_0 \) on \( K \). As before, there is a continuous \( n \)-potential \( p_1 \) such that
1. \( p_1(x) < f_1(x) \) for every \( x \) in \( F \),
2. \( p_1(x) \geq f_1(x) - \epsilon/2 \) for every \( x \) in \( K \),
3. \( \|p_1\|_\infty \leq \|f_1\|_K \|q\|_\infty \).

Now, \( f_1(x) \leq g_1(x) \leq f(x) - p_0(x) \) on \( F \), and \( f_1(x) = g_1(x) = f(x) - p_0(x) \) on \( K \).

Therefore, the above inequalities can be rewritten as follows.
1. \( p_0(x) + p_1(x) < f(x) \) for all \( x \) in \( F \),
2. \( p_0(x) + p_1(x) \geq f_0(x) - \epsilon/2 \) for all \( x \) in \( K \),
3. \( \|p_0\|_\infty \leq \|f_0\|_K \|q\|_\infty \) and \( \|p_1\|_\infty \leq \|f_1\|_K \|q\|_\infty \).

Note that \( \|f_1\|_K \leq \epsilon/2 \).

Proceeding by induction, we get the sequence \( \{p_m\} \), \( m = 0, 1, 2, \ldots \), of bounded continuous \( n \)-potentials and a sequence \( \{f_m\} \), \( m = 0, 1, 2, \ldots \), of continuous functions such that
1. \( \sum_{i=0}^m p_i < f \) on \( F \) for every \( m \),
2. \( \sum_{i=0}^m p_i \geq f_0 - \epsilon/2^m \) on \( K \) for every \( m \),
3. \( \|p_m\|_\infty \leq \|f_m\|_K \|q\|_\infty \) for \( m = 0, 1, 2, \ldots \).

Note that \( \|f_m\|_K < \epsilon/2^m \) for \( m = 1, 2, 3, \ldots \).

Set \( p = \sum_{i=0}^\infty p_m \) on \( \Omega \). Then, it is clear that \( p \) is an \( n \)-superharmonic function and that \( p \leq f \) on \( F \) and \( p = f_0 \) on \( K \). By an analogue of Proposition 2.2.2 of [2], \( p \) is an \( n \)-potential on \( \Omega \). As \( \|p_m\|_\infty \leq \|f_m\|_K \|q\|_\infty \leq \|f_1\|_K \|q\|_\infty \leq (\epsilon/2^m) \|q\|_\infty \) for \( m \geq 1 \), \( \sum_{m=0}^\infty p_m(x) \) converges uniformly on \( \Omega \). Since each \( p_m \) is a continuous function, \( p \) is continuous on \( \Omega \), completing the proof.

The following corollary is an immediate consequence of the above theorem.

**Corollary 13.** (1) Every real-valued continuous function on \( K \) is the restriction to \( K \) of the difference of two positive continuous \( n \)-potentials on \( \Omega \).

(2) Every positive lower semicontinuous function on \( K \) is the restriction to \( K \) of an \( n \)-potential.

(3) If the constant function 1 is \( n \)-superharmonic on \( \Omega \), then every real-valued continuous function on \( K \) is the restriction to \( K \) of an \( n \)-superharmonic function on \( \Omega \).

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