CONTINUOUS FUNCTIONS ON MULTIPOLAR SETS

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Abstract. Let \( \Omega = \Omega_1 \times \cdots \times \Omega_n \) \((n > 1)\) be a product of \( n \) Brelot harmonic spaces each of which has a bounded potential, and let \( K \) be a compact subset of \( \Omega \). Then, \( K \) is an \( n \)-polar set with the property that every \( i \)-section \((1 \leq i < n)\) of \( K \) through any point in \( \Omega \) is \((n - i)\) polar if and only if every positive continuous function on \( K \) can be extended to a continuous potential on \( \Omega \). Further, it has been shown that if \( f \) is a nonnegative continuous function on \( \Omega \) with compact support, then \( MRf \), the multireduced function of \( f \) over \( \Omega \), is also a continuous function on \( \Omega \).

1. Introduction. Let \( \Omega_j \) for \( j = 1, 2, \ldots, n \) be locally compact spaces with countable basis for the topology and be Brelot spaces [5]. The principal results of this paper (cf. Theorem 3 and Theorem 12) characterize certain exceptional compact sets \( K \) contained in the product space \( \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \) in terms of extendibility of positive continuous functions on \( K \), to a positive \( n \)-potential on the entire space. This is a natural generalization of the corresponding results in [7] in the case of a single harmonic space. A key result, independent of interest, needed in the proof of the main results in Theorem 5 shows that the multireduced function of a continuous function is continuous. The results of this paper formed a part of my Ph. D. thesis submitted to McGill University [6]. I would like to thank Professor Gowrisankaran for his help in preparation of this work.

We shall use the notation and results from [3, 4 and 8] concerning \( n \)-harmonic, \( n \)-superharmonic functions, and \( n \)-potentials. We assume that there is a bounded \( n \)-superharmonic function on \( \Omega_1 \times \cdots \times \Omega_n \). Notice that it is equivalent to the assumption that each \( \Omega_j \) \((j = 1, \ldots, n)\) has a bounded potential. If constants are superharmonic, then this assumption trivially holds. Throughout this paper, unless it is explicitly mentioned, \( n \) is an integer \( \geq 2 \), and we denote \( \Omega_1 \times \cdots \times \Omega_n \) by \( \Omega \).

In the course of proving our main results we need a number of results that are routine generalizations of similar ones in the single variable case. The proofs of some of them are not quite obvious, and can be found in [6].

Definition 1. Let \( f \) be an extended real-valued function on \( \Omega \). Define \( MRf(x) = \inf\{u(x): u \text{ is } n\text{-hyperharmonic, and } u \geq f \text{ on } \Omega\} \). For each subset \( E \) of \( \Omega \), let \( X_E \) be the characteristic function of \( E \), and let \( X_E \cdot f \) be the pointwise product function on \( \Omega \). The function \( MR(X_E \cdot f) \) is called the multireduced function of \( f \) over \( E \).
Observe that \((MR(X_E \cdot f))^\dagger\), the lower semicontinuous regularization (see [1]) of 
\(MR(X_E \cdot f)\), is an \(n\)-hyperharmonic function on \(\Omega\).

Let us recall the definition of a section from [8].

**Definition 2.** Let \(E\) be a subset of \(\Omega\), \(n > 1\), and \(i\) an integer such that 
\(1 \leq i \leq n - 1\). Let \(r_1, \ldots, r_i\) be integers with \(1 \leq r_1 < r_2 < \cdots < r_i \leq n\). Let \(t = (x_{r_1}, \ldots, x_{r_i})\) be a point in 
\(\Omega_{r_1} \times \cdots \times \Omega_{r_i}\) \((1 \leq j \leq n)\) with \(\{r_1, \ldots, r_i\} \subseteq \{s_1, \ldots, s_j\} \subseteq \{1, 2, \ldots, n\}\). Then the \((r_1, \ldots, r_i)\)-section of \(E\) through \(t\) is defined as

\[
\begin{align*}
z \text{ in } \prod_{k=1}^{n} \Omega_k, \quad (y_1, \ldots, y_n) & \text{ is in } E, \text{ whenever } y_k = x_k \text{ for } k \text{ in } \{r_1, \ldots, r_i\}, \\
k \text{ not in } \{r_1, \ldots, r_i\}, \quad y_k = z_k & \text{ for } k \text{ not in } \{r_1, \ldots, r_i\},
\end{align*}
\]

which would be denoted by \(E[(r_1, \ldots, r_i), t]\). For \(x\) in \(\Omega\), an \(i\)-section of \(E\) through \(x\) is always denoted by \(E[(r_1, \ldots, r_i), x]\) for some \((r_1, \ldots, r_i)\).

2. **A characterization of a class of multipolar sets.** The following theorem gives a 
necessary condition for a compact set to be \(n\)-polar. We recall that a set \(E\) is \(n\)-polar 
if there exists a positive \(n\)-superharmonic function which is identically infinity on \(E\).

**Theorem 3.** Let \(K\) be a compact subset of \(\Omega\) such that every positive continuous 
function on \(K\) can be uniformly approximated on \(K\) by positive \(n\)-superharmonic 
functions on \(\Omega\). Then,

1. If \(K\) has more than one point, then \(K\) is an \(n\)-polar set.
2. In addition, if point sets are polar in \(\Omega_j\) for \(j = 1, 2, \ldots, n\), then
   a. Every \(i\)-section of \(K\) through any point in \(\Omega\) is \((n - i)\)-polar for \(i = 1, 2, \ldots, n - 1\).
   b. Given \(z\) in \(\Omega\) such that \(z\) not in \(K\), there is a positive \(n\)-superharmonic function \(u\) on 
      \(\Omega\) such that \(u(z) < \infty\) and \(u(x) = \infty\) for all \(x\) in \(K\).

**Proof.** (1) The proof of the fact that the hypothesis implies the \(n\)-polarity of \(K\) is 
very similar to the proof of Theorem 1 of [7].

(2.a) This part is proved by induction on \(n\). Let us make the induction assumption 
that for all products of Brelot spaces \(\Omega_1 \times \cdots \times \Omega_m\) with \(m < n\), the hypothesis of 
the theorem implies the property (a).

Let us now consider a compact set \(K\) contained in \(\Omega\) satisfying the hypothesis. Fix 
\(z\) in \(\Omega\), and \(i\) such that \(1 \leq i \leq n - 1\). By a suitable rearrangement, if necessary, we 
may consider the \(i\)-section \(K[(1, 2, \ldots, i), z]\) to be the general \(i\)-section of \(K\). This 
set is compact and contained in \(\Omega_{i+1} \times \cdots \times \Omega_n\). Let us consider further the 
nontrivial case when \(K[(1, 2, \ldots, i), z]\) contains more than one point. We shall show 
that this set is \((n - i)\)-polar. By the induction hypothesis, it suffices to prove that an 
arbitrary positive continuous function \(f\) on \(K[(1, 2, \ldots, i), z]\) can be uniformly 
approximated on \(K[(1, 2, \ldots, i), z]\) by positive \((n - i)\)-superharmonic functions.

Let us consider the copy \(L[i, z]\) of the above compact set in \(K\), viz., 
\(L[i, z] = \{((z_1, \ldots, z_i, y_{i+1}, \ldots, y_n) \text{ in } K\} \). \(L[i, z]\) is a compact subset of \(K\), and \(f\) can be 
considered as a positive continuous function on this set. By the Tietze extension 
theorem, there exists a positive continuous function \(g\) on \(K\) which extends \(f\). By
hypothesis, for \( \varepsilon > 0 \), there exists a positive \( n \)-superharmonic function \( u \) on \( \Omega \) such that \( |u - g| < \varepsilon \) on \( K \) uniformly. Set

\[
v(y_{i+1}, \ldots, y_n) = u(z_1, \ldots, z_i, y_{i+1}, \ldots, y_n).
\]

This function is \((n - i)\) superharmonic on \( \Omega_{i+1} \times \cdots \times \Omega_n \), and clearly \(|f - v| < \varepsilon\) uniformly on \( K \left[ (1, 2, \ldots, i), z \right] \). Hence, by the induction assumption we conclude that \( K \left[ (1, 2, \ldots, i), z \right] \) is \((n - i)\)-polar. This concludes the proof of (2.a).

(2.b) By Theorem 2.4 of [8], for \( n \)-polar sets, the properties (2.a) and (2.b) are equivalent. This completes the proof of the theorem.

Using the results of [3], it is easy to prove the following proposition.

**Proposition 4.** Let \( v \) be an \( n \)-superharmonic function on \( \Omega \), and let \( \delta_j \) be a regular domain in \( \Omega_i \), \( i = 1, 2, \ldots, n \). For \( j = 0, 1, 2, \ldots, n \), define \( v_j \) on \( \Omega \) as follows.

\[
v_0(x_1, \ldots, x_n) = v(x_1, \ldots, x_n),
\]

and for \( j = 1, 2, \ldots, n \), let

\[
v_j(x_1, \ldots, x_n) = \begin{cases} \int v_{j-1}(x_1, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_n) \, dp_j^\delta(x) & \text{if } x_j \text{ in } \delta_j, \\ v_{j-1}(x_1, \ldots, x_n) & \text{if } x_j \text{ not in } \delta_j. \end{cases}
\]

Then,

1. \( v_j \) is an \( n \)-superharmonic function on \( \Omega \), and \( v_j(x) \leq v_{j-1}(x) \) for all \( x \) in \( \Omega \).
2. \( v_n \) is an \( n \)-harmonic function on \( \delta_1 \times \cdots \times \delta_n \).
3. \( v_n(x) = v(x) \) for every \( x \) in \( (\Omega_1 \setminus \delta_1) \times \cdots \times (\Omega_n \setminus \delta_n) \).

The following theorem is a generalization, to product of harmonic spaces, of a similar result in a single harmonic space (see Proposition 2.2.3 of [2]).

**Theorem 5.** If \( f \) is a nonnegative continuous function on \( \Omega \) with compact support, then \( MRf \) is a continuous function on \( \Omega \).

Recall that the support of a real-valued function is the smallest closed set outside which the function is identically zero.

To prove this theorem we need several lemmas.

**Lemma 6.** Let \( I \) be an indexing set, and \( \{ g_i : i \in I \} \) a family of continuous functions on \( \Omega \) with compact supports. Let there be an \( \varepsilon > 0 \) such that \(|g_i(x) - g_j(x)| \leq \varepsilon\) for all \( x \) in \( \Omega \), and for all \( i \) and \( j \) in \( I \). Further, let us suppose that one of the following two holds:

1. The constant function \( 1 \) is \( n \)-superharmonic on \( \Omega \).
2. There is a relatively compact open set \( U \) of \( \Omega \) such that support of \( g_i \) is contained in \( U \) for every \( i \) in \( I \).

Then, there is a constant \( c > 0 \) such that \(|MRg_i(x) - MRg_j(x)| < ce\) for \( x \) in \( \Omega \), and for \( i \) and \( j \) in \( I \). In case (1) holds, \( c \) can be chosen to be 1, and in the case where (2) holds, \( c \) depends only upon \( U \) (not on the functions \( g_i \)).

**Proof.** If condition (1) holds the proof is trivial. Suppose that (2) holds. Choose a nonnegative continuous function \( f \) on \( \Omega \) with compact support and \( f = 1 \) on \( U \). Then, \( MRf \) is a bounded \( n \)-potential on \( \Omega \), and hence there is a \( c > 0 \) such that
MRf \leq c \text{ on } \Omega. \text{ Observing the fact that } g_i \leq g_j + \epsilon f \text{ on } \Omega \text{ for } i \text{ and } j, \text{ the result follows, and the proof is complete.}

**Lemma 7.** Let \( f \) be a continuous function on \( \Omega \) with compact support. Let \( 1 \leq j < n \) be fixed. Further, let \( d \) be a metric on \( \Omega_1 \times \cdots \times \Omega_j \). For \( x' \) in \( \Omega_1 \times \cdots \times \Omega_j \), define \( f_{x'} \) on \( \Omega_{j+1} \times \cdots \times \Omega_n \) as \( x'' \mapsto f(x', x'') \). Then given \( \epsilon > 0 \), there is an \( \eta > 0 \) such that \( |MRf_{x'}(z) - MRf_{y'}(z)| < \epsilon \), for every \( z \) in \( \Omega_{j+1} \times \cdots \times \Omega_n \) and for all \( x', y' \) in \( \Omega_1 \times \cdots \times \Omega_j \), with \( d(x', y') < \eta \). (Note that the multireduced functions \( MRf_{x'} \) and \( MRf_{y'} \) are defined with respect to the space \( \Omega_{j+1} \times \cdots \times \Omega_n \).)

**Proof of Theorem 5.** The proof is by induction on \( n \). For \( n = 1 \), the result is true (see Proposition 2.2.3 of [2]). Let us prove the result for the case \( n = 2 \). Set \( v = MRf \). Then, it is obvious that \( v \) is lower semicontinuous on \( \Omega \). Therefore, it suffices to prove the upper semicontinuity of \( v \) by proving that for any \((z_1, z_2)\) in \( \Omega_1 \times \Omega_2 \)

\[
\limsup_{(x_1, x_2) \to (z_1, z_2)} v(x_1, x_2) \leq v(z_1, z_2).
\]

Let \((z_1, z_2)\) in \( \Omega_1 \times \Omega_2 \) be fixed, and let \( \epsilon > 0 \). Since \( f \) is continuous, and \( v \) is lower semicontinuous, there are relatively compact open neighborhoods \( U_1 \) and \( U_2 \) of \((z_1, z_2)\) with \( U_1 \subset U_2 \), and there is a 2-harmonic function \( h \) on \( \Omega_1 \times \Omega_2 \) satisfying the following conditions.

1. \( h(z_1, z_2) = 1 \).
2. For all \((x_1, x_2)\) in \( U_1 \subset U_2 \), we have

\[
\begin{align*}
(1) & \quad h(x_1, x_2) \geq 1 - \epsilon, \\
(2) & \quad f(x_1, x_2) \leq (f(z_1, z_2) + \epsilon) h(x_1, x_2),
\end{align*}
\]

and

\[
(3) \quad v(x_1, x_2) \geq (v(z_1, z_2) - \epsilon) h(x_1, x_2).
\]

Let \( d_i \) be a metric in \( \Omega_i \) (\( i = 1, 2 \)). Then, by Lemma 7, there is an \( \eta > 0 \) such that, \( Rf_{x_i}(s) - \epsilon < Rf_{y_i}(s) < Rf_{x_i}(s) + \epsilon \) for all \( s \) in \( \Omega_1 \), and

\[
Rf_{x_i}(t) - \epsilon < Rf_{y_i}(t) < Rf_{x_i}(t) + \epsilon \quad \text{ for all } t \text{ in } \Omega_2,
\]

\[
\text{if } d_1(x_1, y_1) < \eta, \text{ and } d_2(x_2, y_2) < \eta.
\]

Now, choose \( \delta_i \), a regular domain in \( \Omega_i \) (\( i = 1, 2 \)), such that \((z_1, z_2)\) is in \( \delta_1 \times \delta_2 \subset \overline{\delta}_1 \times \overline{\delta}_2 \subset U_2 \). We may assume that the diameter of \( \delta_i \) is small enough and it satisfies

\[
\int d\rho_{x_i}^\delta(t) \geq 1 - \epsilon \quad \text{ for every } x_i \text{ in } \delta_i, \quad i = 1, 2.
\]
Put \( w = v + 2h \). Then, \( w \) is a 2-superharmonic function on \( \Omega_1 \times \Omega_2 \). Define a function \( u \) as follows.

\[
\begin{cases} 
  \iint w(y_1, y_2) \, d\rho_{y_1}^{\delta_1}(y_1) \, d\rho_{y_2}^{\delta_2}(y_2) & \text{if } (x_1, x_2) \text{ in } \delta_1 \times \delta_2, \\
  \int w(x_1, y_2) \, d\rho_{x_2}^{\delta_2}(y_2) & \text{if } x_1 \text{ not in } \delta_1, \text{ and } x_2 \text{ in } \delta_2, \\
  \int w(y_1, x_2) \, d\rho_{x_1}^{\delta_1}(y_1) & \text{if } x_1 \text{ in } \delta_1, \text{ and } x_2 \text{ not in } \delta_2, \\
  w(x_1, x_2) & \text{otherwise.}
\end{cases}
\]

It is clear that \( u \) is a 2-superharmonic function on \( \Omega_1 \times \Omega_2 \), \( u \leq w \), and \( u \) is 2-harmonic on \( \delta_1 \times \delta_2 \).

We claim that \( u \geq (1 - \epsilon)f \) on \( \Omega_1 \times \Omega_2 \). The proof of this claim is given by splitting into four cases, according to whether \( x_1 \) is in \( \delta_1 \) or not, and \( x_2 \) is in \( \delta_2 \) or not.

**Case (i).** Let \( (x_1, x_2) \) be in \( \delta_1 \times \delta_2 \). Then from the definition of \( u \) and \( w \), we have

\[
u(x_1, x_2) = \int w(y_1, y_2) \, d\rho_{y_1}^{\delta_1}(y_1) \, d\rho_{y_2}^{\delta_2}(y_2) \\
\geq \int (v(z_1, z_2) + \epsilon)h(y_1, y_2) \, d\rho_{y_1}^{\delta_1}(y_1) \, d\rho_{y_2}^{\delta_2}(y_2) \\
\geq (v(z_1, z_2) + \epsilon)h(x_1, x_2) \\
\geq (1 - \epsilon)f(x_1, x_2) \ (\text{as } f > 0),
\]

**Case (ii).** Let \( x_1 \) be in \( \delta_1 \) and \( x_2 \) not in \( \delta_2 \). Then,

\[
u(x_1, x_2) = \int w(y_1, x_2) \, d\rho_{y_1}^{\delta_1}(y_1) \\
= \int (Rf_{x_1}(x_2) - \epsilon) + 2eh(x_1, x_2) \, d\rho_{y_1}^{\delta_1}(y_1) \\
\geq (Rf_{x_1}(x_2) - \epsilon)(1 - \epsilon) + 2eh(x_1, x_2) \ (\text{as } Rf_{x_1} > f_{x_1} \text{ on } \Omega_2) \\
= (f(x_1, x_2) - \epsilon)(1 - \epsilon) + 2\epsilon(1 - \epsilon) \ (\text{using (1)}) \\
> (1 - \epsilon)f(x_1, x_2) \ (\text{as } f > 0).
\]
Case (iii). Let $x_1$ not be in $\delta_1$ and $x_2$ in $\delta_2$. The proof is similar to the previous case.

Case (iv). Let $x_1$ not be in $\delta_1$ and $x_2$ not in $\delta_2$. Then the proof trivially follows from the definitions of $u$, $v$ and $w$. Thus, the claim is proved.

Now, $u \geq (1 - \epsilon)f$ on $\Omega_1 \times \Omega_2$ gives that $u \geq (1 - \epsilon)v$ on $\Omega_1 \times \Omega_2$. In particular, if $(x_1, x_2)$ is in $\delta_1 \times \delta_2$ then

$$
(1 - \epsilon)u(x_1, x_2) \leq \int \int v(y_1, y_2) \, d\rho_{x_1}^{\delta_1}(y_1) \, d\rho_{x_2}^{\delta_2}(y_2) + 2h(x_1, x_2).
$$

The right-hand side of the above inequality is a 2-harmonic function on $\delta_1 \times \delta_2$, hence is a continuous function on $\delta_1 \times \delta_2$. Taking lim sup as $(x_1, x_2) \to (z_1, z_2)$ in $\Omega_1 \times \Omega_2$, and noticing that $v$ is a 2-superharmonic function we get

$$
(1 - \epsilon) \limsup_{(x_1, x_2) \to (z_1, z_2)} v(x_1, x_2) \leq v(z_1, z_2) + 2\epsilon.
$$

As $\epsilon > 0$ is arbitrary, we have

$$
\limsup_{(x_1, x_2) \to (z_1, z_2)} v(x_1, x_2) \leq v(z_1, z_2).
$$

Since $(z_1, z_2)$ is an arbitrary point in $\Omega_1 \times \Omega_2$, it follows that $v$ is upper semicontinuous on $\Omega_1 \times \Omega_2$. Hence, $v$ is continuous on $\Omega_1 \times \Omega_2$, and this concludes the proof for the case $n = 2$.

To complete the proof of the induction, we proceed from the case of functions of $n - 1$ variables to functions of $n$ variables in exactly the same way. We remark that the choice of $u$ in the above proof is replaced by $w_n$ as defined in Proposition 4. The rest of the details are absolutely the same. This allows us to conclude that $MRf$ is in general a continuous function whenever $f$ is a nonnegative continuous function with compact support, completing the proof of the theorem.

As an immediate consequence, we have the following corollary.

**Corollary 8.** If $f$ is a nonnegative continuous function on $\Omega$ with compact support, then $MRf$ is a continuous $n$-potential on $\Omega$.

Though the following result is essentially a corollary to the above theorem, we will state it as a theorem due to its importance. We omit the proof.

**Theorem 9.** Let $v$ be a positive $n$-superharmonic function on $\Omega$. Then, there is a sequence $v_j$ of continuous $n$-potentials such that $v_j$ increases pointwise to $v$ on $\Omega$ as $j \to \infty$.

From now on $K$ is a compact $n$-polar subset of $\Omega$, such that every $i$-section of $K$ through any point of $\Omega$ is $(n - i)$-polar, for $i = 1, 2, \ldots, n - 1$.

The next theorem is the converse of Theorem 3.

**Theorem 10.** Given a positive continuous function $f$ on $K$ and an $\epsilon > 0$, there exists a continuous $n$-potential $p$ on $\Omega$ such that $|f - p| < \epsilon$ on $K$. 

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Proof. Let \( \{U_j\} \), \( j = 1, 2, 3, \ldots \), be a decreasing sequence of relatively compact subsets of \( \Omega \) such that \( U_j \supseteq \overline{U}_{j+1} \supseteq U_{j+1} \) for \( j = 1, 2, \ldots \), and \( K = \cap\{U_j: j = 1, 2, 3, \ldots \} \). For each \( j \), let \( f_j \) be a nonnegative continuous extension of \( f \) to \( \Omega \) with support of \( f_j \subset U_j \). By taking infimum at each stage, we may assume that \( \{f_j\} \) is a decreasing sequence of functions on \( \Omega \).

Put \( p_j = MRf_j \). Then, by Corollary 8, \( p_j \) is a continuous \( n \)-potential, for each \( j \). Following the proof Theorem 4 of [7], we can show that \( p_j \) decreases pointwise to \( f \) on \( K \) as \( j \to \infty \). Using Dini's Theorem, we conclude that \( p_j \) converges to \( f \) uniformly on \( K \). Thus, there is an \( m \) such that \( |f(x) - p_j(x)| < \epsilon \) if \( j \geq m \), for all \( x \) in \( K \). The choice \( p = p_m \) meets the requirement of the theorem, completing the proof.

The following result is an analogue of Theorem 5 of [7], and can be proved analogously with the help of Theorem 10.

Proposition 11. Let \( f_0 \) be a positive continuous function on \( K \), and \( F_0 \) be a relatively compact open neighborhood of \( K \). Put \( F = F_0 \), and let \( f \) be a nonnegative continuous extension of \( f_0 \) to \( \Omega \), such that \( f > 0 \) on \( F \). Then, given \( \epsilon > 0 \), there exists a continuous potential \( p \) on \( \Omega \) such that \( p < f \) on \( F \), and \( p \geq f_0 - \epsilon \) on \( K \).

Our ultimate aim is the following theorem, for the case \( n \geq 2 \).

Theorem 12. Given a positive continuous function \( f_0 \) on \( K \), there is a continuous \( n \)-potential \( p \) on \( \Omega \) such that \( p = f_0 \) on \( K \).

Proof. The existence of an \( n \)-potential \( p \) such that \( p = f_0 \) on \( K \) can be proved as in the case \( n = 1 \). (See Theorem 2 of [7].) However, in proving the continuity of \( p \), in the case \( n = 1 \), we have explicitly used the fact that \( Rg \) is harmonic outside the support of \( g \). This result is no longer valid for \( MRg \) when \( n > 1 \). Hence, we modify the proof as follows. We also note that the same method works in the case \( n = 1 \).

Let \( \epsilon > 0 \). For a continuous function \( g \) on \( K \), define
\[
\|g\|_K = \sup\{|g(x)|: x \in K\},
\]
and if \( g \) is a bounded continuous function on \( \Omega \), then define
\[
\|g\|_\infty = \sup\{|g(x)|: x \in \Omega\}.
\]
Let \( q \) be a bounded continuous \( n \)-potential on \( \Omega \). We may assume that \( q \geq 1 \) on \( K \).

Let \( F_0 \) be a relatively compact open set containing \( K \) and let \( F = \overline{F_0} \). Choose \( f \) a nonnegative continuous extension of \( f_0 \) to \( \Omega \) with \( f > 0 \) on \( F \), and \( \|f\|_\infty = \|f_0\|_K \). Then, by the previous theorem, there is a continuous \( n \)-potential \( q_0 \) on \( \Omega \) such that \( q_0 < f \) on \( F \) and \( q_0 > f_0 - \epsilon \) on \( K \). Let \( p_0 = \inf\{\|f_0\|_K q, q_0\} \) on \( \Omega \). Then, \( p_0 \) is a bounded continuous \( n \)-potential on \( \Omega \). Further, \( p_0 \leq q_0 < f \) on \( F \). If \( x \) is in \( K \), then \( q(x) \geq 1 \), and hence,
\[
\|f_0\|_K q(x) \geq \|f_0\|_K = \|f\|_\infty \geq f(x) > q_0(x).
\]
Therefore, \( p_0 = q_0 \) on \( K \) and hence, \( p_0 \geq f_0 - \varepsilon \) on \( K \). Thus, there is a bounded continuous \( n \)-potential \( p_0 \) such that

1. \( p_0(x) < f(x) \) for all \( x \) in \( F \),
2. \( p_0(x) \geq f_0(x) - \varepsilon \) for all \( x \) in \( K \),
3. \( \|p_0\|_\infty < \|f_0\|_\infty \|q\|_\infty \).

Put \( g_1 = \max(f - p_0, 0) \) on \( \Omega \). Then, \( g_1 \) is a nonnegative continuous function and \( g_1 > 0 \) on \( F \). Let \( g_2 \) be a nonnegative continuous on \( \Omega \) such that \( g_2 = g_1 \) on \( K \) and \( g_2 > 0 \) on \( F \). Since \( g_2 = g_1 = f - p_0 \) on \( K \), we may even choose \( g_2 \) such that \( \|g_2\|_\infty = \|f - p_0\|_\infty \). Set \( f_1 = \inf\{g_1, g_2\} \). Then, \( f_1 \) is a nonnegative continuous function on \( \Omega \) with \( f_1 > 0 \) on \( F \) and \( f_1 = f - p_0 \) on \( K \). As before, there is a continuous \( n \)-potential \( p_1 \) such that

1. \( p_1(x) < f_1(x) \) for every \( x \) in \( F \),
2. \( p_1(x) \geq f_1(x) - \varepsilon/2 \) for every \( x \) in \( K \),
3. \( \|p_1\|_\infty \leq \|f_1\|_\infty \|q\|_\infty \).

Now, \( f_1(x) \leq g_1(x) \leq f(x) - p_0(x) \) on \( F \), and \( f_1(x) = g_1(x) = f(x) - p_0(x) \) on \( K \).

Therefore, the above inequalities can be rewritten as follows.

1. \( p_0(x) + p_1(x) < f(x) \) for all \( x \) in \( F \),
2. \( p_0(x) + p_1(x) \geq f_0(x) - \varepsilon/2 \) for all \( x \) in \( K \),
3. \( \|p_0\|_\infty \leq \|f_0\|_\infty \|q\|_\infty \) and \( \|p_1\|_\infty \leq \|f_1\|_\infty \|q\|_\infty \).

Note that \( \|f_1\|_\infty \leq \varepsilon/2 \).

Proceeding by induction, we get the sequence \( \{p_m\}, m = 0, 1, 2, \ldots, \) of bounded continuous \( n \)-potentials and a sequence \( \{f_m\}, m = 0, 1, 2, \ldots, \) of continuous functions such that

1. \( \Sigma_{i=0}^m p_i < f \) on \( F \) for every \( m \),
2. \( \Sigma_{i=0}^m p_i > f_0 - \varepsilon/2^m \) on \( K \) for every \( m \),
3. \( \|p_m\|_\infty \leq \|f_m\|_\infty \|q\|_\infty \) for \( m = 0, 1, 2, \ldots \).

Note that \( \|f_m\|_K < \varepsilon/2^m \) for \( m = 1, 2, 3, \ldots \).

Set \( p = \Sigma_{i=0}^\infty p_m \) on \( \Omega \). Then, it is clear that \( p \) is an \( n \)-superharmonic function and that \( p \leq f \) on \( F \) and \( p \geq f_0 \) on \( K \). By an analogue of Proposition 2.2.2 of [2], \( p \) is an \( n \)-potential on \( \Omega \). As \( \|p_m\|_\infty \leq \|f_m\|_K \|q\|_\infty \leq \varepsilon/2^m \|q\|_\infty \) for \( m \geq 1 \), \( \Sigma_{m=0}^\infty p_m(x) \) converges uniformly on \( \Omega \). Since each \( p_m \) is a continuous function, \( p \) is continuous on \( \Omega \), completing the proof.

The following corollary is an immediate consequence of the above theorem.

**Corollary 13.** (1) Every real-valued continuous function on \( K \) is the restriction to \( K \) of the difference of two positive continuous \( n \)-potentials on \( \Omega \).

(2) Every positive lower semicontinuous function on \( K \) is the restriction to \( K \) of an \( n \)-potential.

(3) If the constant function 1 is \( n \)-superharmonic on \( \Omega \), then every real-valued continuous function on \( K \) is the restriction to \( K \) of an \( n \)-superharmonic function on \( \Omega \).

**Bibliography**


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