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MAXIMAL COMPACT NORMAL SUBGROUPS

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ABSTRACT. The main concern is the existence of a maximal compact normal subgroup K in a locally compact group G, and whether or not G/K is a Lie group. G has a maximal compact subgroup if and only if G/G_0 has. Maximal compact subgroups of totally disconnected groups are open. If the bounded part of G is compactly generated, then G has a maximal compact normal subgroup K and if $B(G)$ is open, then G/K is Lie. Generalized FC-groups, compactly generated type I IN-groups, and Moore groups share the same properties.

Introduction. We shall be concerned with the conditions under which a locally compact group G possesses a maximal compact normal subgroup K and whether or not G/K is a Lie group. It is known that locally compact connected groups and compactly generated Abelian groups have maximal compact normal subgroups with Lie factor groups. Moreover, almost connected groups (i.e., when G/G_0 is compact, where G_0 is the identity component of G) and compactly generated FC-groups share these results.

For general locally compact Abelian groups, the corresponding assertion is not true—even if the group is σ-compact. Consider any Ω_a of a-adic numbes; Ω_a is a locally compact σ-compact Abelian group which has no maximal compact (normal) subgroup.

In this paper we show that a locally compact group G has a maximal compact subgroup if and only if G/G_0 has (Theorem 1). This generalizes substantially the corresponding result for almost connected groups mentioned before. It also reduces the question of the existence of maximal compact subgroups to that of totally disconnected groups—in which we show every maximal compact subgroup is open (Corollary 1 of Theorem 2). In [**1**] it is proved that a generalized FC-group G has a maximal compact normal subgroup K. Here we show that G/K is a Lie group (Theorem 5). We also prove that, if the bounded part of G, $B(G)$, is compactly generated, then G has a maximal compact normal subgroup (Theorem 6). Particularly, we see that compactly generated Moore groups, type I IN-groups, and compact extensions of compactly generated nilpotent groups have the desired properties (Theorems 7 and 8). Finally some sufficient conditions for G/K to be a Lie group are summarized in Theorem 10.

If a topological group G has a maximal compact subgroup N, then clearly gNg^{-1} is a maximal compact subgroup of G for any $g \in G$. A maximal compact subgroup contains every compact normal subgroup. Hence $K = \bigcap_{g\in G} gNg^{-1}$ is the maximal compact normal subgroup of G.

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THEOREM 1. *Let G be a locally compact group. Then G has maximal compact subgroups if and only if G/G_0 has.*

PROOF. Suppose G has a maximal compact subgroup K. The image of K under the natural mapping $\varphi: G \to G/G_0$ is the compact subgroup KG_0/G_0. Suppose there is a compact subgroup M/G_0, where M is a closed subgroup of G containing G_0, such that $KG_0/G_0 \subset M/G_0$. Since $M_0 = G_0$, M/M_0 is compact. Let N be a compact normal subgroup of M such that M/N is a Lie group [**5**, p. 175]. Then M/N has a finite number of connected components and K/N is a maximal compact subgroup of M/N. So $M/N = (K/N)(G_0N/N)$ by [**6**, Theorem 3.1]. It follows that $M = KG_0$, i.e., KG_0/G_0 is maximal in G/G_0.

Conversely, suppose G/G_0 has a maximal compact subgroup M/G_0. Using Theorem 3.1 of [**6**] and the previous argument, let K be a maximal compact subgroup of M and $M = KG_0$. If K' is a compact subgroup of G and $K' \supsetneqq K$, then $M/G_0 \subsetneqq K'G_0/G_0$. This contradicts the maximality of M/G_0 in G/G_0. Therefore K is a maximal compact subgroup of G.

The mere assumption that G/G_0 has a maximal compact subgroup and hence G has a maximal compact normal subgroup K does not imply that G/K is a Lie group [**1**, Example 2].

Another aspect of Theorem 1 is that, in a sense, now one needs only to determine when a totally disconnected group has a maximal compact subgroup. We show that such a subgroup, if it exists, is open.

THEOREM 2. *In a locally compact totally disconnected group G, every compact subgroup is contained in a compact open subgroup of G.*

PROOF. Let K be a compact subgroup of G, and let H be an open compact subgroup of G. There is a neighborhood V of e such that $kVk^{-1} \subset H$ for all $k \in K$. Choose a compact open subgroup $N \subset V$, and let M be the closure of the subgroup generated by $\bigcup_{k\in K} kNk^{-1}$. Then M is a compact subgroup, KM is a compact open subgroup of G, and $K \subset KM$.

COROLLARY 1. *In a locally compact totally disconnected group G, every maximal compact subgroup is open.*

COROLLARY 2. *If G is a locally compact group and K is a maximal compact subgroup of G, then KG_0 is open.*

PROOF. KG_0/G_0 is a maximal compact subgroup of the totally disconnected group G/G_0. Since KG_0/G_0 is open in G/G_0, KG_0 is open in G.

DEFINITIONS. (1) An element $g \in G$ is called *bounded* if the conjugacy class $\{xgx^{-1}\colon x \in G\}$ has a compact closure. The set of all bounded elements of G is denoted by $B(G)$. A topological group is an *FC-group* if $G = B(G)$.

(2) An element $g \in G$ is called *periodic* if it is contained in a compact subgroup of G. The set of all periodic elements of G is denoted by $P(G)$.

(3) A locally compact group G is a *generalized FC-group* of length n if there exists a finite sequence $A_1, A_2, \ldots, A_{n+1}$ of closed normal subgroups of G such that $G = A_1 \supset A_2 \supset \cdots \supset A_{n+1} = \{e\}$ and each factor A_i/A_{i+1} is a compactly generated FC-group.

(4) A locally compact group G is *pro-Lie* if every neighborhood of the identity contains a compact normal subgroup K such that G/K is a Lie group.

LEMMA 3. *Let G be a locally compact group and N a closed normal subgroup of G. If K and K' are maximal compact normal subgroups of G and N respectively, then $K' = K \cap N$. In particular, if $K \subseteq N$, then $K = K'$.*

PROOF. $K \cap N$ is a compact normal subgroup of N. Hence $K \cap N \subseteq K'$. It is now sufficient to prove that K' is a normal subgroup of G. Given any $g \in G$, $gK'g^{-1}$ is a compact subgroup of N; and for $n \in N$,

$$n(gK'g^{-1})n^{-1} = g(g^{-1}ng)K'(g^{-1}ng)^{-1}g^{-1} = gK'g^{-1}$$

since $g^{-1}ng \in N$. It follows that $gK'g^{-1}$ is a compact normal subgroup of N. Therefore $gK'g^{-1} \subseteq K'$ and this shows that K' is normal in G. The lemma is thus proved.

LEMMA 4. *Let G be a locally compact group which has a maximal compact normal subgroup K. Let H be a closed normal compactly generated subgroup of G which has a maximal compact normal subgroup P and H/P is a pro-Lie group. If G/H is compact, then G/K is a Lie group.*

PROOF. By Lemma 3, P is a normal subgroup of G. Let $G' = G/P$, $H' = H/P$, and $K' = K/P$. Now $G/H \cong G'/H'$, thus G'/H' is compact, where H' is a closed normal compactly generated pro-Lie group. By Theorem 4 of [**3**], G' is also pro-Lie. But $G'/K' \cong G/K$, hence G/K is a pro-Lie group with no nontrivial compact normal subgroup. It follows that G/K is a Lie group.

THEOREM 5. *A generalized FC-group G has a maximal compact normal subgroup K and G/K is a Lie group.*

PROOF. The existence of K is proved in [**1**, Theorem 3]. To prove that the corresponding factor group is Lie, consider the chain $G = A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} = \{e\}$ as in the definition. We proceed by induction, noting that, for $n = 1$, the conclusion follows from [**2**, Theorem 3.20]. We assume that every generalized FC-group with sequence $\{A_i\}$ of length less than or equal to n has the desired property. Thus G/A_n has a maximal compact normal subgroup N/A_n such that $(G/A_n)/(N/A_n) \cong G/N$ is a Lie group, where N is a closed normal subgroup of G containing A_n. As a compactly generated FC-group, A_n has a maximal compact normal subgroup P and A_n/P is a Lie group. Since P is maximal in A_n, it is closed and normal in G. Let $\varphi: G \to G/A_n$ be the natural mapping. Then, $\varphi(K) \subset N/A_n$, so $K \subset N$.

Now N is a generalized FC-group with sequence $N \supset A_n \supset \{e\}$. Thus N has a maximal compact normal subgroup which must be K, by Lemma 3, since $K \subset N$. The groups N, A_n, K, and P satisfy the conditions of Lemma 4; hence N/K is a Lie group. Since G/N is a Lie group and $G/N \cong (G/K)/(N/K)$, G/K is also a Lie group.

COROLLARY 1. *Let G be a locally compact group and H a compactly generated normal FC-subgroup of G. If G/H is compact, then G has a maximal compact normal subgroup K and G/K is a Lie group.*

PROOF. Observe that G/H is a compact FC-group, thus G is a generalized FC-group with sequence $G \supset H \supset \{e\}$.

COROLLARY 2. *Let G be a compactly generated group such that $G/\overline{B(G)}$ is compact. Then G has a maximal compact normal subgroup K and G/K is a Lie group.*

PROOF. Let $H = \overline{B(G)}$. Then H is compactly generated. It follows that H is an FC-group by Proposition 4 of [**8**]. Thus, by Corollary 1, the conclusion is obtained.

We derive another corollary for later use. A locally compact group G is called a *Z-group* if $G/Z(G)$ is compact, where $Z(G)$ is the center of G.

COROLLARY 3. *Let G be a locally compact group and H a compactly generated Z-subgroup of G. If G/H is compact, then G has a maximal compact normal subgroup K and G/K is a Lie group.*

PROOF. Immediate from Corollary 1.

The following example shows that the result of Corollary 1 is not covered already.

EXAMPLE. Let $G = \mathbf{R}^2 \times_\eta T$, where T is the unit circle and acts on $\mathbf{R}^2$ by rotation, i.e., the automorphism of $\mathbf{R}^2$ induced by $e^{i\theta} \in T$ is a rotation of $\mathbf{R}^2$ by the angle θ. If $H = \mathbf{R}^2$, then H is a compactly generated FC-group and G/H is compact. Since $(x, y, 0)(0, 0, \pi)(-x, -y, 0) = (2x, 2y, \pi)$, the conjugacy class of $(0, 0, \pi)$ contains a nonrelatively compact set. Thus G is not an FC-group. But G is a Lie group with the identity as the maximal compact normal subgroup.

Let G be locally compact. Then $B(G) \cap P(G)$ is a normal subgroup generated by all compact normal subgroups of G [**7**, Corollary 5.6]. Hence, if K is the maximal compact normal subgroup of G, then $K = P(G) \cap B(G)$.

THEOREM 6. *If G is a locally compact group and $B(G)$ is compactly generated, then G has a maximal compact normal subgroup.*

PROOF. $\overline{B(G)}$ is compactly generated and hence by Proposition 4 of [**8**], it is an FC-group. Thus $\overline{B(G)}$ has a maximal compact normal subgroup K. As in the proof of Lemma 3, we can see that K is normal in G. Suppose K is not maximal. Then there is a compact normal subgroup N of G such that $K \subset N$ and $K \neq N$. But $N \subset P(G) \cap B(G) \subset \overline{B(G)}$. This contradicts the maximality of K in $\overline{B(G)}$. It follows that K is the maximal compact normal subgroup of G.

The converse of this theorem does not hold. In Example 1 of [**1**], $B(G) = \sum H_i \times \{1\}$ is not compactly generated. This theorem can be applied in abstract group theory:

COROLLARY. *Let G be a group. If $B(G)$, the set of elements of G which have a finite conjugacy class, is finitely generated, then G has a maximal finite normal subgroup.*

If $B(G)$ is compactly generated and open in G, then G is an IN-group by Theorem 1 [**8**]. Thus G/K is a Lie group.

The next several results, which establish sufficient conditions for a group to have a maximal compact normal subgroup, are expressed in terms of representations of a group.

DEFINITIONS. Let G be a locally compact group.

(1) G is called a *Moore group* if G has only finite dimensional irreducible (continuous unitary) representations.

(2) G is called a *type* I *group* if the von Neumann algebra generated by any representation of G is type I.

THEOREM 7. *Let G be a compactly generated type* I IN-*group. Then G has a maximal compact normal subgroup K and G/K is a Lie group.*

PROOF. It is known that a compactly generated group has a compact normal subgroup N such that $G' = G/N$ is second countable. Since $G'/B(G')$ is finite by Proposition 4.1 of [**4**], it follows that G' has a maximal compact normal subgroup M, by Corollary 2 of Theorem 5. Then $K = \varphi^{-1}(M)$ is the maximal compact normal subgroup of G, where φ is the natural mapping of G onto G'. Now G/K is a Lie group since G is an IN-group.

As a result, every compactly generated type I MAP- or SIN-group shares the conclusion of Theorem 7. The same assertion follows for compactly generated Moore groups, since a Moore group is a type I SIN-group.

THEOREM 8. *Let G be a locally compact group. Let H be a compactly generated nilpotent normal subgroup of G such that G/H is compact. Then G has a maximal compact normal subgroup K and G/K is a Lie group.*

PROOF. Since G is pro-Lie by [**3**, Theorem 9], we only need to show the existence of K. For this purpose, we use induction on the nilpotency class n of H. For $n = 1$, H is Abelian and the conclusion follows from Corollary 3 of Theorem 5. Suppose the assertion is true for groups of class less than n. By [**3**, Proposition 8], $Z(H)$ is compactly generated. Clearly $H/Z(H)$ is compactly generated with nilpotency class less than n. Thus $G/Z(H)$ has a maximal compact normal subgroup $M/Z(H)$, where M is a closed normal subgroup of G. Then $G/M \cong (G/Z(H))/(M/Z(H))$ has no nontrivial compact normal subgroup. Hence it is sufficient to show that M has a maximal compact normal subgroup. But this follows from Corollary 3 of Theorem 5, since M and $Z(H)$ satisfy its hypotheses. Hence the proof is complete.

On many occasions we find the following lemma very useful.

LEMMA 9. *Let G be a locally compact group. Suppose*

(1) *G has a maximal compact normal subgroup K,*

(2) *N is a σ-compact closed normal subgroup of G and has a maximal compact normal subgroup K_1 such that N/K_1 is a Lie group, and*

(3) *G/N is a Lie group.*

Then G/K is a Lie group.

PROOF. Observe that $K_1 = K \cap N$. Since N is σ-compact, $KN/K \cong N/K \cap N$. It follows that KN/K is a Lie group. Also, G/KN is a Lie group since G/N is a Lie group. Hence G/K is a Lie group since $G/KN \cong (G/K)/(KN/K)$.

THEOREM 10. *Let G be a locally compact group which has a maximal compact normal subgroup K. Let H be a closed normal subgroup of G such that G/H has a compact normal subgroup with Lie factor group. If H satisfies one of the following conditions, then G/K is a Lie group.*

(I) *H is σ-compact and compact extensions of H have maximal compact normal subgroups with Lie factor groups.*

(II) *H is compactly generated nilpotent.*

(III) *H is almost connected.*

PROOF. We use Lemma 9 to prove this theorem. If N/H is a compact normal subgroup of G/H with Lie factor group, then G/N is a Lie group and hence conditions (1) and (3) of Lemma 9 are satisfied for (I), (II), and (III). Condition (2) of Lemma 9 is satisfied trivially for (I); is satisfied for (II) using Theorem 8; and is satisfied for (III) since N is almost connected.

The compactly generated condition on H in (II) seems necessary. In [**1**, Example 2], $K = \{e\}$, $H = \sum H_i$, and G/H is compact. The maximal compact normal subgroup of H is the identity and H is a Lie group, but G/K is not a Lie group.

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