JULIA’S LEMMA AND WOLFF’S THEOREM
FOR J*-ALGEBRAS
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ABSTRACT. Julia’s lemma and Wolff’s theorem are established for (Fréchet-)
holomorphic maps of bounded symmetric homogeneous domains in infinite
dimensional complex Banach spaces called J*-algebras.

Julia [8, p. 87] to matrix-valued holomorphic maps of a complex variable. Next,
algebras and for holomorphic maps of proper contraction operators in the sense of
functional calculus, respectively.

In another paper [4], K. Fan extended the classical theorem of J. Wolff [14]
on iterates of self-maps to holomorphic maps of proper contraction operators in
the sense of functional calculus. Similar extensions of Wolff’s theorem to (Fréchet-)
holomorphic maps of the open unit ball and the generalized upper half-plane in C^N
were given by G. N. Chen [1]. Furthermore, Y. Kubota [9] and B. D. MacCluer
[10], using different methods, proved some results on iterates of Wolff-Denjoy type
[14, 2] in C^N.

The main results of this paper are of the above two types (see §2). We first
prove a general version of Julia’s lemma for (Fréchet-)holomorphic maps of bounded
symmetric homogeneous domains in infinite dimensional complex Banach spaces
called J*-algebras. We next prove, as an application of this, the extension of Wolff’s
theorem. In particular, our results imply Julia’s lemma and Wolff’s theorem for
arbitrary complex Hilbert spaces (see §5), B*-algebras, C*-algebras, and ternary
algebras.

\mathcal{L}(H, K) denote the Banach space of all bounded linear operators X from H to K
with the operator norm, and let \mathfrak{A} \subset \mathcal{L}(H, K) be a J*-algebra (see L. A. Harris
[7]), i.e. a normed complex linear subspace of \mathcal{L}(H, K) closed under the operation
X \mapsto XX^*.

Let
\begin{align*}
\mathfrak{A}_0 &= \{X \in \mathfrak{A} : \|X\| < 1\}, \\
\mathfrak{A}_1 &= \{X \in \mathfrak{A} : \|X\| \leq 1\},
\end{align*}

and, for X \in \mathfrak{A}_0, Z \in \mathfrak{A}_1, and \alpha \geq 1, let
\begin{align*}
W_Z(X) &= (I_H - Z^*X)A_X^{-1}(I_H - X^*Z), \\
c_\alpha(Z, X) &= \{I_H \alpha \|W_Z(X)\| + ZZ^*\}^{-1},
\end{align*}

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JULIA'S LEMMA AND WOLFF'S THEOREM

\[ r_\alpha(Z, X) = \|c_\alpha(Z, X)\|^{1/2} \|I_H \{\alpha\|W_Z(X)\| - 1\} + Z^*c_\alpha(Z, X)Z\|^{1/2}, \]

where \( I_H \) is the identity map on \( H \) and \( A_X = I_H - X^*X \).

We shall use these notations in proving the following Julia-type lemma for \( \mathfrak{A} \).

**Lemma 2.1.** Let \( F: \mathfrak{A}_0 \rightarrow \mathfrak{A}_0 \) be a holomorphic map in \( \mathfrak{A}_0 \). If \( \{z_m\} \subset \mathfrak{A}_0 \) is such that

\[ \lim_{m \to \infty} \|Z_m - V\| = 0 \]

and

\[ \lim_{m \to \infty} \|F(Z_m) - V\| = 0 \]

for some \( V \in \mathfrak{A}_1 \setminus \mathfrak{A}_0 \), and if

\[ \lim_{m \to \infty} \frac{\|A_F(z_m)\|}{1 - \|z_m\|^2} = \alpha \neq \infty, \]

then

\[ \|X - c_1(V, X)V\| \leq r_1(V, X), \]

\[ \|W_V[F(X)]\| \leq \alpha\|W_V(X)\|, \]

and

\[ \|F(X) - c_\alpha(V, X)V\| \leq r_\alpha(V, X) \]

hold for all \( X \in \mathfrak{A}_0 \).

We can also use Lemma 2.1 to obtain a Wolff-type theorem for \( \mathfrak{A} \).

**Theorem 2.2.** Let \( F: \mathfrak{A}_0 \rightarrow \mathfrak{A}_0 \) be a compact holomorphic map having no fixed point in \( \mathfrak{A}_0 \) and let \( F^{[n]} \) denote the \( n \)th iterate of \( F \) (i.e., \( F^{[1]} = F \) and \( F^{[n]} = F \circ F^{[n-1]} \) for \( n \geq 2 \)). Then there exist \( \{Z_m\} \subset \mathfrak{A}_0 \) and \( V \in \mathfrak{A}_1 \setminus \mathfrak{A}_0 \) such that

\[ \lim_{m \to \infty} \|Z_m - V\| = 0 \quad \text{and} \quad F(V) = V. \]

Moreover, \( \|X - c_1(V, X)V\| \leq r_1(V, X) \) for all \( X \in \mathfrak{A}_0 \), and if

\[ \lim_{m \to \infty} \frac{\|A_{F(z_m)}\|}{1 - \|z_m\|^2} = \alpha \neq \infty, \]

then

\[ \|W_V[F^{[n]}(X)]\| \leq \alpha\|W_V(X)\| \]

and

\[ \|F^{[n]}(X) - c_\alpha(V, X)V\| \leq r_\alpha(V, X) \]

hold for all \( x \in \mathfrak{A}_0 \) and \( n = 1, 2, \ldots \).

For the special case when \( \mathfrak{A} = H \), see §5.

**Lemma 3.1.** If $F: \mathbb{A}_0 \rightarrow \mathbb{A}_0$ is a holomorphic map in $\mathbb{A}_0$, then

\[
\|W_F(Z)[F(X)]\| \leq \frac{\|A_F(Z)\|}{1 - \|Z\|^2} \|W_Z(X)\|
\]

for all $X, Z \in \mathbb{A}_0$.

**Proof.** Let $X, Z \in \mathbb{A}_0$. It follows from [13, Theorem 1(a)] that

\[
\|A_F^{-1/2}W_F(Z)[F(X)]A_F^{-1/2}\| \leq \|A_Z^{-1/2}W_Z(X)A_Z^{-1/2}\|.
\]

But

\[
\|A_Z^{-1/2}\|^2 = \|A_Z^{-1}\| = (1 - \|Z\|^2)^{-1}
\]

and

\[
I_H\|A_F^{-1/2}\|^{-1} \leq A_F^{-1/2} \quad \text{since} \quad I_H \leq A_F^{-1/2}.
\]

Thus, using (3.3) and (3.4), from (3.2) we get (3.1). This completes the proof.

**Lemma 3.2.** If $X \in \mathbb{A}_0$, $Z \in \mathbb{A}_1$, and $D$ satisfy

\[
\|W_Z(X)\| \leq D,
\]

then

\[
\|X - (I_H D + ZZ^*)^{-1}Z\|
\]

\[
\leq \|(I_H D + ZZ^*)^{-1}\|^{1/2}\|I_H(D - 1) + Z^*(I_H D + ZZ^*)^{-1}Z\|^{1/2}.
\]

**Proof.** Inequality (3.5) implies

\[
\|A_X^{-1/2}(I_H - X^*Z)(I_H - Z^*X)A_X^{-1/2}\| \leq D
\]

or, equivalently,

\[
(I_H - X^*Z)(I_H - Z^*X) \leq A_X D.
\]

Further, the above inequality is identical to

\[
\{X^* - Z^*(I_H D + ZZ^*)^{-1}\}(I_H D + ZZ^*)\{X - (I_H D + ZZ^*)^{-1}Z\}
\]

\[
\leq I_H(D - 1) + Z^*(I_H D + ZZ^*)^{-1}Z.
\]

Consequently,

\[
\|(I_H D + ZZ^*)^{1/2}\{X - (I_H D + ZZ^*)^{-1}Z\}\|^2 \leq \|I_H(D - 1) + Z^*(I_H D + ZZ^*)^{-1}Z\|.
\]

Hence (3.6) follows. This completes the proof.

Now, we assume that $\{Z_m\} \subset \mathbb{A}_0$ satisfies (2.1)–(2.3). Then, from Lemma 3.1 it follows that (2.5) holds, and (2.4) and (2.6) follow from the relations $\|W_Z(X)\| = D_1$ and $\|W_V[F(X)]\| \leq D_\alpha$, $D_\alpha = \alpha\|W_V(X)\|$, respectively, if we use Lemma 3.2.
4. Proof of Theorem 2.2.

**Lemma 4.1.** If \( F : \mathbb{A}_0 \rightarrow \mathbb{A}_0 \) is a compact holomorphic map having no fixed point in \( \mathbb{A}_0 \), then there exist \( \{ Z_m \} \subset \mathbb{A}_0 \) and a fixed point \( V \in \mathbb{A}_1 \setminus \mathbb{A}_0 \) of \( F \) such that

\[
\lim_{m \to \infty} ||Z_m - V|| = 0.
\]

**Proof.** Let \( F_m = \alpha_m F \), \( 0 < \alpha_m < 1 \), \( m = 1, 2, \ldots \), and let \( \lim_{m \to \infty} \alpha_m = 1 \). Let \( \{ Z_m \} \subset \mathbb{A}_0 \) be such that \( F_m(Z_m) = Z_m \), \( m = 1, 2, \ldots \); such a sequence exists by the Earle-Hamilton theorem [3]. Since \( F(\mathbb{A}_0) \) is contained in a compact subset of \( \mathbb{A} \), \( F \) has no fixed points in \( \mathbb{A}_0 \) and \( F(Z_m) = Z_m/\alpha_m \), \( m = 1, 2, \ldots \), we may assume that \( \lim_{m \to \infty} ||Z_m - V|| = 0 \) and \( \lim_{m \to \infty} ||F(Z_m) - V|| = \lim_{m \to \infty} ||Z_m/\alpha_m - V|| = 0 \) for some \( V \in \mathbb{A}_1 \setminus \mathbb{A}_0 \). This completes the proof.

Let now \( X \in \mathbb{A}_0 \) and let \( F_m^{[n]} \) denote the \( n \)th iterate of \( F_m \), \( m = 1, 2, \ldots \). By Lemma 4.1 and (3.1),

\[
(4.1) |W_{Z_m}[F_m^{[n]}(X)]| \leq \frac{||A_{Z_m}||}{1 - ||Z_m||^2} ||W_{Z_m}(X)||.
\]

Furthermore (see [12, formula (4.6), p. 158]),

\[
(4.2) \lim_{m \to \infty} ||F_m^{[n]}(X) - F_m^{[n]}(X)|| = 0.
\]

Thus, if (2.7) holds, from (4.1) and (4.2) we obtain (2.8). By Lemma 3.2, inequality (2.8) implies (2.9).

5. Concluding remarks. Let

\[
H_0 = \{ x \in H : ||x|| < 1 \}, \quad H_1 = \{ x \in H : ||x|| \leq 1 \},
\]

and, for \( x \in H_0, v \in H_1 \setminus H_0 \), and \( \alpha \geq 1 \), let

\[
c_\alpha(v, x) = \frac{1 - ||x||^2}{\alpha|1 - \langle x, v \rangle|^2 + 1 - ||x||^2},
\]

\[
r_\alpha(v, x) = \frac{\alpha|1 - \langle x, v \rangle|^2}{\alpha|1 - \langle x, v \rangle|^2 + 1 - ||x||^2}.
\]

Identifying \( H \) with \( \mathcal{L}(\mathbb{C}, H) \), as corollaries from our main results and their proofs we get the following Julia's lemma and Wolff's theorem for arbitrary complex Hilbert spaces.

**Lemma 5.1.** Let \( F : H_0 \rightarrow H_0 \) be a holomorphic map in \( H_0 \). If \( \{ z_m \} \subset H_0 \) is such that

\[
\lim_{m \to \infty} ||z_m - v|| = 0 \quad \text{and} \quad \lim_{m \to \infty} ||F(z_m) - v|| = 0
\]

for some \( v \in H_1 \setminus H_0 \), and if

\[
\lim_{m \to \infty} \frac{1 - ||F(z_m)||^2}{1 - ||z_m||^2} = \alpha \neq \infty,
\]

then

\[
||x - c_1(v, x)v|| = r_1(v, x), \quad \frac{|1 - \langle F(x), v \rangle|^2}{1 - ||F(x)||^2} \leq \alpha \frac{|1 - \langle x, v \rangle|^2}{1 - ||x||^2},
\]

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and
\[ \|F(x) - c_\alpha(v, x)v\| \leq r_\alpha(v, x) \]
for all \( x \in H_0 \).

**Theorem 5.2.** Let \( F: H_0 \to H_0 \) be a compact holomorphic map having no fixed points in \( H_0 \) and let \( F^{[n]} \) denote the \( n \)th iterate of \( F \). Then there exists \( v \in H_1 \setminus H_0 \) such that \( F(v) = v \),
\[ \|x - c_1(v, x)v\| = r_1(v, x), \]
\[ \frac{|1 - \langle F^{[n]}(x), v \rangle|^2}{1 - \|F^{[n]}(x)\|^2} \leq \frac{|1 - \langle x, v \rangle|^2}{1 - \|x\|^2}, \]
and
\[ \|F^{[n]}(x) - c_1(v, x)v\| \leq r_1(v, x) \]
for all \( x \in H_0 \) and \( n = 1, 2, \ldots \).

**Remarks.** If \( H = \mathbb{C} \), Lemma 5.1 and Theorem 5.2 imply the results of Julia [8, p. 87] and Wolff [14], respectively.

**References**


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