ABSTRACT. Let \( f(z) = z + az^2 + \cdots \) be analytic in the unit disc \( U \) and let \( k(z) = z/(1 - z) \). The classic Marx-Strohhäcker result, that a convex (univalent) function \( f \) is starlike of order \( \frac{1}{2} \), can be written in terms of differential subordinations as
\[
zf''(z)/f'(z) < zk''(z)/k'(z) \Rightarrow zf'(z)/f(z) < zk'(z)/k(z).
\]
The authors determine general conditions on \( k \) for which this relation holds. They also determine a different set of general conditions on \( k \) for which
\[
zf'(z)/f(z) < zk'(z)/k(z) \Rightarrow f(z)/z < k(z)/z.
\]
Finally, differential subordinations with starlike superordinate functions are considered.

1. Introduction. Let \( f \) be analytic in the unit disc \( U = \{z : |z| < 1\} \). The function \( f \), with \( f'(0) \neq 0 \), is convex (univalent) if and only if \( \text{Re}[zf''(z)/f'(z) + 1] > 0 \) in \( U \). The function \( f \), with \( f(0) = 0 \) and \( f'(0) \neq 0 \), is starlike (univalent) if and only if \( \text{Re}[zf'(z)/f(z)] > 0 \) in \( U \). If, in addition, \( \text{Re}[zf'(z)/f(z)] > \frac{1}{2} \) then \( f \) is said to be starlike of order \( \frac{1}{2} \) [1, Vol. 1, p. 137].

The classic result of Marx [3] and Strohhäcker [10] asserts that a convex function is starlike of order \( \frac{1}{2} \), that is
\[
\text{Re}[zf''(z)/f'(z) + 1] > 0, \quad z \in U \Rightarrow \text{Re}[zf'(z)/f(z)] > \frac{1}{2}, \quad z \in U.
\]
We can simplify this relation by expressing it in terms of subordination. If \( F \) and \( G \) are analytic in \( U \), then \( F \) is subordinate to \( G \), written \( F < G \) or \( F(z) \subset G(z) \), if 
\[
G \text{ is univalent, } F(0) = G(0) \text{ and } F(U) \subset G(U).
\]
If we let \( k(z) = z/(1 - z) \), then (1) can be rewritten as
\[
zf''(z)/f'(z) < zk''(z)/k'(z) \Rightarrow zf'(z)/f(z) < zk'(z)/k(z).
\]
This Marx-Strohhäcker differential subordination system of first type holds for other functions \( k \); in [5, p. 194] and in [7, Corollary 2.3] it is shown that (2) also holds when \( k(z) = z/(1 - z^2) \) is the Koebe function. In 2 of this article we determine general conditions on \( k \) for which (2) holds.

If \( k \) is the Koebe function, then Marx [3] and Strohhäcker [10] also showed that
\[
zf'(z)/f(z) < zk'(z)/k(z) \Rightarrow f(z)/z < k(z)/z.
\]
§3 deals with general first order differential subordinations of the form \( \psi(p(z),zp'(z)) < h(z) \), where \( h \) is a starlike function. In addition, as a special application of these results we determine conditions on \( k \) for which the Marx-Strohhäcker differential subordination system of second type, as given in (3), holds.

We close this section with two lemmas that will be used in subsequent sections.

**Lemma 1.** Let \( F \) be analytic in \( U \) and let \( G \) be analytic and univalent on \( U \), with \( F(0) = G(0) \). If \( F \) is not subordinate to \( G \), then there exist points \( z_0 \in U \) and \( \zeta_0 \in \partial U \), and an \( m > 1 \) for which \( F(|z| < |z_0|) \subset G(U) \),

- (i) \( F(z_0) = G(\zeta_0) \),
- (ii) \( z_0 F'(z_0) = m\zeta_0 G'(\zeta_0) \).

The proof of a more general form of this lemma may be found in [4, Lemma 1].

The second lemma deals with the concept of a subordination chain. A function \( L(z,t) \), \( z \in U \), \( t \geq 0 \), is a subordination chain if \( L(-,t) \) is analytic and univalent in \( U \) for all \( t \geq 0 \), \( L(z,-) \) is continuously differentiable on \([0,\infty)\) for all \( z \in U \), and \( L(z,t_1) < L(z,t_2) \) when \( 0 \leq t_1 < t_2 \).

**Lemma 2** [8, p. 159]. The function \( L(z,t) = a_1(t)z + \cdots \), with \( a_1(t) \neq 0 \) for all \( t \geq 0 \), is a subordination chain if and only if \( \Re[z(\partial L/\partial z)/(\partial L/\partial t)] > 0 \) for \( z \in U \) and \( t \geq 0 \).

2. Marx-Strohhäcker differential subordination systems of first type.

**Theorem 1.** Let \( q \) be univalent in \( U \), with \( q(0) = 1 \). Set \( Q(z) = zq'(z)/q(z) \), \( h(z) = q(z) + Q(z) \) and suppose that

- (i) \( Q \) is starlike in \( U \) (\( \log q \) is convex in \( U \)), and
- (ii) \( \Re[z h'(z)/Q(z)] = \Re[q(z) + z Q'(z)/Q(z)] > 0 \), \( z \in U \).

If \( B \) is an analytic function on \( U \) such that

\[
B(z) < q(z) + z q'(z)/q(z) = h(z),
\]

then the analytic solution \( p \) of the differential equation

\[
zp'(z) + B(z)p(z) = 1 \quad (p(0) = 1)
\]

satisfies \( p < 1/q \).

**Proof.** Conditions (i) and (ii) imply that the function \( h \) is univalent (close-to-convex) [1, Vol. 2, p. 2], \( h(0) = 1 \), and hence (4) is well defined with \( B(0) = 1 \). If we let \( b(z) = \int_0^z [B(t) - 1]t^{-1} \, dt \), then (5) has an analytic solution given by \( p(z) = \int_0^z e^{b(t)} \, dt/z e^{b(z)} \). Condition (i) implies that \( q(z) \neq 0 \) in \( U \). Hence the function \( 1/q \) is analytic and univalent in \( U \).

Without loss of generality we can assume \( q \) is univalent and \( q(z) \neq 0 \) on \( \overline{U} \). If not, then we can replace \( q \), \( B \), and \( p \) by \( q_r(z) = q(rz) \), \( B_r(z) = B(rz) \), and \( p_r(z) = p(rz) \) respectively, where \( 0 < r < 1 \). These new functions satisfy the conditions of the theorem on \( \overline{U} \). We would then prove \( p_r < 1/q_r \), and by letting \( r \to 1^- \) we obtain \( p < 1/q \).

The function

\[
L(z,t) = h(z) + tQ(z) = q(z) + (1 + t)z q'(z)/q(z)
\]
is analytic in $U$ for all $t \geq 0$, and it is continuously differentiable on $[0, \infty)$ for all $z \in U$. Since $Q'(0) = q'(0) \neq 0$ we have $\partial L/\partial z(0,t) = q'(0)(2 + t) \neq 0$ for $t \geq 0$. From (ii) we obtain
\[
\Re \left[ z \frac{\partial L}{\partial z} / \frac{\partial L}{\partial t} \right] = \Re \left[ q(z) + (1 + t) \frac{zQ'(z)}{Q(z)} \right] \geq \Re \frac{zQ'(z)}{Q(z)} > 0,
\]
for $z \in U$ and $t \geq 0$. Hence by Lemma 2, $L(z, t)$ is a subordination chain and so we have $L(z, t_1) < L(z, t_2)$ for $0 \leq t_1 \leq t_2$. From (6) we have $L(z, 0) = h(z)$ and so we obtain
\[
(7) \quad L(z, t) \not\in h(U)
\]
for $|z| = 1$ and $t \geq 0$.

Now assume that $p \not< 1/q$. From Lemma 1 there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$, and an $m \geq 1$ such that $p(z_0) = 1/q(\zeta_0)$ and $z_0p'(z_0) = -m\zeta_0q'(\zeta_0)/q^2(\zeta_0)$. Using these results together with (5) and (6) we obtain
\[
B(z_0) = \frac{1}{p(z_0)} - \frac{z_0p'(z_0)}{p(z_0)} = q(\zeta_0) + \frac{m\zeta_0q'(\zeta_0)}{q(\zeta_0)} = L(\zeta_0, m - 1).
\]
By using (7) we deduce that $B(z_0) \not\in h(U)$, which contradicts (4). Hence $p < 1/q$, completing the proof of the theorem.

We now turn our attention to considering the Marx-Strohhäcker differential subordination system of first type as given in (2). If we let $p(z) = zf'(z)/f(z)$ and $q(z) = zk'(z)/k(z)$ then (2) becomes
\[
p(z) + zp'(z)/p(z) < q(z) + zq'(z)/q(z) \Rightarrow p(z) < q(z).
\]
This differential subordination result was considered in [5, Theorem 3]. Before using this general result to prove (2), however, we have to first prove that $zf'(z)/f(z)$ is analytic in $U$, a condition which does not follow directly from the left side of (2).

**THEOREM 2.** Let $q$ satisfy the conditions of Theorem 1 and let
\[
k(z) = z \exp \int_0^z \left[ q(t) - 1 \right] t^{-1} dt.
\]
If $f(z) = z + a_2z^2 + \cdots$ is analytic in $U$, and
\[
zf''(z)/f'(z) < zk''(z)/k'(z),
\]
then $zf'(z)/f(z)$ is analytic in $U$ and $zf'(z)/f(z) < zk'(z)/k(z)$.

**PROOF.** From (8) we obtain
\[
q(z) = \frac{zk'(z)}{k(z)} \quad \text{and} \quad h(z) = q(z) + \frac{zq'(z)}{q(z)} = 1 + \frac{zk''(z)}{k'(z)}.
\]
Thus $zk'(z)/k(z)$ is univalent, and by (ii) of Theorem 1 we see that $zk''(z)/k'(z)$ is also univalent. Since condition (9) implies $f'(z) \neq 0$, the function $B(z) = 1 + zf''(z)/f'(z)$ is analytic in $U$, and satisfies $B < h$. For this particular $B$ equation (5) has the analytic solution $p(z) = f(z)/zf'(z)$. Since all the conditions of Theorem 1 are satisfied, we have $p < 1/q$. Since $1/q \not< 0$ this implies that $p \neq 0$, and so $1/p(z) = zf'(z)/f(z)$ is analytic in $U$. In addition, from $p < 1/q$ and $1/q \neq 0$ we obtain $1/p < q$, which yields the desired conclusion $zf'(z)/f(z) < zk'(z)/k(z)$.
EXAMPLES. (a) The function \( q(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \), with \( 0 \leq \alpha < 1 \), satisfies (i) and (ii) of Theorem 1. From Theorem 2 we obtain \( k(z) = k(\alpha, z) = \frac{z}{(1 - z)^{2(1 - \alpha)}} \) and
\[
\frac{zf''(z)}{f'(z)} < \frac{zk''(\alpha, z)}{k'(\alpha, z)} = \frac{2(1 - \alpha)(2 + (1 - 2\alpha)z)}{(1 - z)(1 + (1 - 2\alpha)z)}
\]
\[
\Rightarrow \frac{zf'(z)}{f(z)} < \frac{zk'(\alpha, z)}{k(\alpha, z)} = \frac{1 + (1 - 2\alpha)z}{1 - z}.
\]
The cases \( \alpha = \frac{1}{2} \) and \( \alpha = 0 \) correspond to the functions \( z/(1 - z) \) and \( z/(1 - z)^2 \) respectively, which were mentioned in §1.

(b) The function \( q(z) = \frac{z}{1 - e^{-z}} \), which is a convex (univalent) function [6, p. 70] satisfies \( q(0) = 1 \) and
\[
Q(z) = zq'(z)/q(z) = 1 + z/(1 - e^z) = 1 - q(-z).
\]
Since \( q \) is convex we conclude that \( Q \) is also convex. By using (1) we obtain \( \text{Re} zQ'(z)/Q(z) > \frac{1}{2} \), which implies condition (i) of Theorem 1. Condition (ii) also holds since
\[
\text{Re}[q(z) + zQ'(z)/Q(z)] \geq \frac{1}{(e - 1) + 1/2} > 0.
\]
Hence by Theorem 2 we obtain \( k(z) = e^z - 1 \) and
\[
\frac{zf''(z)}{f'(z)} < \frac{zk''(z)}{k'(z)} = z
\]
\[
\Rightarrow \frac{zf'(z)}{f(z)} < \frac{zk'(z)}{k(z)} = z/(1 - e^{-z}).
\]

(c) The function \( q(z) = [(1 + z)/(1 - z)]^\beta \), with \( 0 < \beta \leq 1 \), satisfies (i) and (ii) of Theorem 1 and by Theorem 2 we have
\[
\frac{zf''(z)}{f'(z)} < \left[\frac{1 + z}{1 - z}\right]^\beta + \frac{z^2 + 2\beta z - 1}{1 - z^2} \Rightarrow \frac{zf'(z)}{f(z)} < \left[\frac{1 + z}{1 - z}\right]^\beta.
\]

(d) Employing the function \( q(z) = e^{\lambda z} \) in Theorem 1 we obtain \( Q(z) = \lambda z \) and \( q(z) + zQ'(z)/Q(z) = e^{\lambda z} + 1 \). Condition (i) is satisfied and condition (ii) will be satisfied if \( \text{Re}[e^{\lambda z} + 1] > 0 \). A simple calculation shows that this condition holds for \( |\lambda| \leq r_0 = 2.183... \), where \( r_0 = y_0/\sin y_0 \) and \( y_0 \) is the smallest root of \( \cos y + \exp(y/\COT y) = 0 \). Hence by Theorem 2, for \( |\lambda| \leq r_0 = 2.183... \), we obtain \( k(z) = z \exp \int_0^z (e^{\lambda t} - 1)t^{-1} dt \) and
\[
\frac{zf''(z)}{f'(z)} < \frac{zk''(z)}{k'(z)} = e^{\lambda z} + \lambda z - 1
\]
\[
\Rightarrow \frac{zf'(z)}{f(z)} < \frac{zk'(z)}{k(z)} = e^{\lambda z}.
\]

(e) The function \( q(z) = 1 + \lambda z, |\lambda| \leq 1 \), satisfies the conditions of Theorems 1 and 2 and we obtain \( k(z) = ze^{\lambda z} \) and
\[
\frac{zf''(z)}{f'(z)} < \frac{zk''(z)}{k'(z)} = \frac{\lambda z(2 + \lambda z)}{(1 + \lambda z)} \Rightarrow \frac{zf'(z)}{f(z)} < \frac{zk'(z)}{k(z)} = 1 + \lambda z.
\]
In the special case \( \lambda = 1 \) this simplifies to
\[
zf''(z)/f'(z) < z(2 + z)/(1 + z) \Rightarrow |zf'(z)/f(z) - 1| < 1.
\]
This is an improvement of the result \( |f''(z)/f'(z)| \leq 3/2 \Rightarrow |zf'(z)/f(z) - 1| < 1 \) proved in [7, Theorem 3] and the result \( |zf''(z)/f'(z) + 1| < 2 \Rightarrow |zf'(z)/f(z) - 1| < 1 \) proved in [5, Theorem 5].
3. Differential subordinations with starlike superordinate function.

THEOREM 3. Let \( H(z) = z + a_2z^2 + \cdots \) be starlike in \( U \) and let \( \psi: C^2 \to C \) be a function of the form \( \psi(r, s) = w(r) \cdot s^\alpha \) with \( \alpha > 0 \). In addition, suppose that

\[
\psi(q(z), zq'(z)) = w(q(z)) \cdot (zq'(z))^\alpha = H(z)
\]

has a univalent solution \( q \). If \( p \) is analytic in \( U \), with \( p(0) = q(0) \), then

\[
w(p(z)) \cdot (zp'(z))^\alpha < H(z) \Rightarrow p(z) < q(z).
\]

PROOF. If \( p \not= q \) then by Lemma 1 there exist points \( z_0 \in U \) and \( \xi_0 \in \partial U \), and an \( m > 1 \) such that \( p(z_0) = q(\xi_0) \) and \( z_0p'(z_0) = m\xi_0q'(\xi_0) \). Using these results in (10) we obtain

\[
w(p(z_0)) \cdot (z_0p'(z_0))^\alpha = w(q(\xi_0)) \cdot (m\xi_0q'(\xi_0))^\alpha = m^\alpha H(\xi_0).
\]

Since \( H(U) \) is starlike and \( m^\alpha \geq 1 \) we obtain \( w(p(z_0)) \cdot (z_0p'(z_0))^\alpha \not\in H(U) \). This contradicts (10) and so we must have \( p < q \).

Note that (10) requires \( \psi(q(0), 0) = 0 \). The special case \( \psi(r, s) = r^{1-\alpha}s^\alpha \), with \( 0 < \alpha \leq 1 \), satisfies Theorem 3, with \( p(0) = q(0) \), and was considered in [4, p. 169].

For the purposes of this paper, Theorem 3 is most interesting when \( \psi(r, s) = s/r \). Applying the theorem in this case we obtain

THEOREM 4. Let \( H(z) = z + a_2z^2 + \cdots \) be starlike in \( U \) and suppose that

\[
Q(z) = \exp \int_0^z H(t)t^{-1}\, dt
\]

is univalent. If \( p \) is analytic in \( U \), with \( p(0) = 1 \), then

\[
zp'(z)/p(z) < zQ'(z)/Q(z) = H(z) \Rightarrow p(z) < Q(z).
\]

COROLLARY 4.1. If \( H \) is a convex (univalent) and \( \text{Re}H(z) > -1 \), then \( Q \) as given in (12) is univalent and (13) holds.

PROOF. From (1) and (12) we obtain

\[
\text{Re}[1 + zQ''(z)/Q'(z)] = \text{Re}[H(z) + zH'(z)/H(z)] > -1 + 1 = -\frac{1}{2},
\]

which shows that \( Q \) is univalent [9, Corollary 3].

If we set \( H(z) = q(z) - 1 \) and \( p(z) = f(z)/z \) we obtain

COROLLARY 4.2. Let \( q \) be convex with \( \text{Re}q(z) > 0 \) and let

\[
k(z) = z \exp \int_0^z [q(t) - 1]t^{-1}\, dt.
\]

If \( f(z) = z + a_2z^2 + \cdots \) is analytic in \( U \), then

\[
zf'(z)/f(z) < zk'(z)/k(z) \Rightarrow f(z)/z < k(z)/z.
\]

REMARKS. 1. This relation is precisely the one that the Koebe function satisfied in (3) of §1. The corollary lists sufficient conditions on \( k \) to ensure that the Marx-Strohhäcker differential subordination system of second type holds.
2. The functions $q$ used in Examples (a), (b), (c), and (e) also satisfy the conditions of the corollary, whereas $q$ in (d) does not for $|\lambda| > 1$. As a result, for Example (a) we have

$$\frac{zf''(z)}{f'(z)} < \frac{zk''(\alpha, z)}{k'(\alpha, z)} \Rightarrow \frac{zf'(z)}{f(z)} < \frac{zk'(\alpha, z)}{k'(\alpha, z)} \Rightarrow \frac{f(z)}{z} < \frac{k(\alpha, z)}{z},$$

where $k(\alpha, z) = z/(1 - z)^{2(1 - \alpha)}$ and $0 \leq \alpha < 1$.

For Example (b) we have

$$zf''(z)/f'(z) < z \Rightarrow zf'(z)/f(z) < (1 - e^{-z}) \Rightarrow f(z)/z < (e^z - 1)/z,$$

and for Example (c) we have the extension

$$zf'(z) < \left(1 + z\right) \Rightarrow f(z)/z < \exp\left(\left(1 + t\right)/\left(1 - t\right) - 1\right)t^{-1} dt$$

for $0 < \beta \leq 1$. In the special case $\beta = \frac{1}{2}$ this simplifies to

$$zf'(z)/f(z) < \left(1 + z\right)^{\frac{1}{2}} \Rightarrow f(z)/z < \frac{2 \exp(SIN^{-1}z)}{1 + (1 - z^2)^{1/2}} = Q(z).$$

From the last corollary we know that Re$(1 + zQ''(z)/Q'(z)) > -\frac{1}{2}$ and that $Q$ is univalent. To show that the function $Q$ is also convex it is sufficient to show Re$(1 + zQ''(z)/Q'(z)) \geq 0$ for $|z| = 1$. Since

$$1 + zQ''(z)/Q'(z) = [(1 + z)/(-1 - z)]^{1/2} - 1 + \frac{1}{2}[(1 - z)^{-1} + (1 - z^2)^{-1/2}],$$

if we let $z = e^{it}, 0 \leq t \leq \pi$, we obtain

$$(1 + z)/(1 - z) = \cot(t/2), \quad 1 - z^2 = -2ie^{it}\sin t,$$

and

Re$(1 + zQ''(z)/Q'(z)) = (\frac{1}{2} \cot(t/2))^{1/2} - \frac{3}{4} + [\cot(t/2) + \sin(t/2)]/4(\sin(t))^{1/2}.$

If we let $x = (\cot(t/2))^{1/2} \geq 0$, we obtain

Re$(1 + zQ''(z)/Q'(z)) = (5x^2 - 3\sqrt{2}x + 1)/4\sqrt{2} > 0.$

We close this section by considering another differential subordination having a starlike superordinate function. In [4, p. 170] and in [2, p. 192] it is shown that if $h$ is convex and $p$ is analytic in $U$, then

$$p(z) + zp'(z) < h(z) \Rightarrow p(z) < z^{-1} \int_0^zh(t) dt.$$

We can replace the superordinate function $h$ with a function of the form $[(1 + z)/(1 - z)]^\alpha$. For $\alpha > 1$ this function is not convex, but is starlike with respect to 1.

**Theorem 5.** Let $\beta_0 = 1.21872...$ be the solution of $\beta\pi = 3\pi/2 - TAN^{-1}\beta$ and let $\alpha = \alpha(\beta) = \beta + 2TAN^{-1}\beta/\pi$ for $0 < \beta \leq \beta_0$. If $p$ is analytic in $U$, with $p(0) = 1$, then

$$p(z) +zp'(z) < \left[\frac{1 + z}{1 - z}\right]^{\alpha} \Rightarrow p(z) < \left[\frac{1 + z}{1 - z}\right]^{\beta}.$$
Proof. Note that $\beta \pi \leq 3\pi/2 - \tan^{-1} \beta$ and $\beta \leq 3 - \alpha$ for $0 < \beta \leq \beta_0$. If we let $h(z) = [(1 + z)/(1 - z)]^\alpha$ and $q(z) = [(1 + z)/(1 - z)]^\beta$ then $h(U)$ and $q(U)$ are domains given by the sectors $|\arg h| < \alpha \pi/2$ and $|\arg q| < \beta \pi/2$ respectively. We need to show that $p < q$. If this is not true, then there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ and $p(|z| < |z_0|) \subset q(U)$. We need to consider separately the case $p(z_0) \neq 0$ which corresponds to a point on one of the rays on the sector $q(U)$, and the case $p(z_0) = 0$ which corresponds to the corner of the sector.

Case (i). If $p(z_0) \neq 0$ then $\zeta_0 \neq \pm 1$. If we let $r_i = (1 + \zeta_0)/(1 - \zeta_0)$ and use Lemma 1 we obtain

$$ p(z_0) + z_0 p'(z_0) = q(\zeta_0) + m_\zeta_0 q'(\zeta_0) = (r_i)^\beta - (r_i)^{\beta - 1} m \beta (1 + r^2)/2$$

and

$$\arg (p(z_0) + z_0 p'(z_0)) = \beta \pi/2 + \tan^{-1} m/3(r + 1/r)/2.$$ 

Since $\tan^{-1}$ is an increasing function and $m \geq 1$, by considering the cases $r > 0$ and $r < 0$ we obtain

$$\beta \pi/2 + \tan^{-1} \beta \leq |\arg (p(z_0) + z_0 p'(z_0))| \leq \beta \pi/2 + \pi/2.$$ 

Using the fact that $\beta \pi \leq 3\pi/2 - \tan^{-1} \beta$ for $0 < \beta \leq \beta_0$ and the definition of $\alpha$ we obtain

$$\alpha \pi/2 \leq |\arg (p(z_0) + z_0 p'(z_0))| \leq 2\pi - \alpha \pi/2.$$ 

This implies that $p(z_0) + z_0 p'(z_0)$ lies outside the sector $h(U)$, and since this contradicts the hypothesis of the theorem we must have $p < q$.

Case (ii). The case $p(z_0) = 0$ can occur only if $\beta \geq 1$, since $\beta < 1$ implies that the sector angle of $q(U)$ is less than $\pi$ and $p(|z| = |z_0|)$ cannot pass through such a corner without itself having a corner. If $p'(z_0) \neq 0$, then the smallest possible value of $\arg (z_0 p'(z_0))$ is given by $(2\pi - \beta \pi/2) - \pi/2$, which occurs when $p(|z| = |z_0|)$ is tangent to the lower ray of the sector $q(U)$. The largest possible value of $\arg (z_0 p'(z_0))$ is given by $\beta \pi/2 + \pi/2$, which occurs when $p(|z| = |z_0|)$ is tangent to the upper ray of the sector $q(U)$. Using these limitations and the fact that $\beta \leq 3 - \alpha$ we obtain

$$\alpha \pi/2 \leq \arg (p(z_0) + z_0 p'(z_0)) \leq 2\pi - \alpha \pi/2,$$

which gives the same contradiction as Case (i). The subcase $p'(z_0) = 0$ also yields a contradiction since $p(z_0) + z_0 p'(z_0) = 0 \notin h(U)$.

Remarks. If we take $\beta = 1$ in Theorem 5 we obtain $\alpha(1) = \frac{3}{2}$ and

$$p(z) + z p'(z) < \left[ \frac{1 + z}{1 - z} \right]^{3/2} \Rightarrow p(z) < \frac{1 + z}{1 - z}.$$ 

In this case the angle of the sector $h(U)$ is $270^\circ$. The case with the largest angle corresponds to $\alpha_0 = \alpha(\beta_0) = \alpha(1.218\ldots) = 1.781\ldots$. In this case the angle of the sector $h(U)$ is $320.62\ldots^\circ$, while the angle of sector $q(U)$ is $219.36\ldots^\circ$.

References


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