A SINGULAR SPACE RELATED TO THE POINT-OPEN GAME

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ABSTRACT. The authors construct a completely regular space $X_0$ such that Player I has a winning strategy in the point-open game $G(X_0)$, but $X_0$ has no $\sigma$-closure-preserving cover by $C$-scattered closed sets.

The purpose of this paper is to present the following.

EXAMPLE. A completely regular space $X_0$ such that Player I has a winning strategy in the point-open game $G(X_0)$, but $X_0$ has no $\sigma$-closure-preserving cover by $C$-scattered closed sets.

Given a space $X$, the point-open game $G(X)$ is defined as follows: at move $n$, Player I chooses a point $x_n$ in $X$, and then Player II chooses an open neighborhood $U_n$ of $x_n$. Player I wins the play $(x_0, U_0, x_1, U_1, \ldots)$ of $G(X)$ if $\bigcup\{U_n : n < \omega\} = X$ (see [G, T]). If instead of points $x_n$ Player I chooses compact sets $C_n$, and if Player I has a winning strategy in the resulting game, then $X$ is called compact-like (see [T, T]). If a point $x$ in a space $X$ has an open neighborhood $U$ such that $\overline{U}$ is compact, then $x$ is called a point of local compactness of $X$. A space $X$ such that each nonempty closed subspace has a point of local compactness is called $C$-scattered [T]. A family $\mathcal{F}$ of subsets of $X$ with the property that for any subfamily of $\mathcal{F}$, the closure and the sum commute, is called closure-preserving [M]. A countable union of closure-preserving families is called $\sigma$-closure-preserving.

The space $X_0$ in the above example is a modification of the space constructed by Nogura [N]; however, it improves his result in several aspects. Nogura showed, under the continuum hypotheses, that a compact-like space need not have a countable cover by $C$-scattered closed subsets, answering a question of Telgársky [T, T]. Here, (1) the continuum hypothesis is eliminated, (2) the game condition involving Player I is substantially strengthened, and (3) the countable union is generalized to a $\sigma$-closure-preserving union.

Since each compact space is $C$-scattered, it follows that $X_0$ has no $\sigma$-closure-preserving cover by compact sets. In contrast, a hereditarily metacompact compact-like space does have a closure-preserving cover by compact sets (see [JST]). Moreover, each compact-like space is the union of countably many $C$-scattered subsets (see [T]), where the $C$-scattered subsets cannot be closed in general (it follows from [N] and also from the above example).
CONSTRUCTION. Let $T = \{t_\alpha : \alpha < \omega_1\}$ be a subset of the closed unit interval $[0,1]$, where $t_\alpha \neq t_\beta$ for $\alpha \neq \beta$. Let $T_0 = T \times \{\omega_1\}$ and

$$X = \{(t_\alpha, \beta) : \alpha < \beta < \omega_1\} \cup T_0.$$ 

The space $X$ is a subspace of $[0,1] \times [0,\omega_1]$, where $[0,\omega_1]$ has the standard interval topology (that is, the topology generated by left-open right-closed intervals). The space $X_0$ is obtained from $X$ by contracting the set $T_0$ into the singleton $\{x_0\}$.

CLAIM 1. Player I has a winning strategy in $G(X_0)$.

PROOF. Observe that the complement of any neighborhood of $x_0$ in $X_0$ is countable. Therefore, if the first move of Player I is $x_0$, he can easily take care of the remaining countable set.

For a subset $H$ of $X_0$ and a $t \in T$ define

$$H_t = \{\alpha < \omega_1 : (t, \alpha) \in H\}.$$ 

CLAIM 2. If $H$ is closed in $X_0$, then $H_t$ is closed in $[0,\omega_1)$.

This is immediate, since the map which takes $(t, \alpha)$ to $\alpha$ is a homeomorphism between a subspace of $X_0$ and a closed subspace of $[0,\omega_1)$.

For a subset $H$ of $X_0$ define

$$T_H = \{t \in T : \text{H}_t \text{ is uncountable}\}.$$ 

CLAIM 3. If $H$ is closed in $X_0$, then $T_H$ is closed in $[0,\omega_1)$.

PROOF. Let $t = \lim_{n} t(n)$, where $t \in T$ and $t(n) \in T_H$ for each $n < \omega$. Then each $H_t(n)$ is uncountable and closed in $[0,\omega_1)$, and thus $E = \bigcap \{H_t(n) : n < \omega\}$ is also uncountable and closed in $[0,\omega_1)$. Take an $\alpha < \omega_1$ such that $t_\alpha = t$. Then $E \cap (\alpha,\omega_1) \subset H_t$, and hence $t \in T_H$.

CLAIM 4. If $\mathcal{H}$ is a $\sigma$-closure-preserving family of closed subsets of $X_0$, then

$$\bigcup \{T_H : H \in \mathcal{H}\} = T_{\cup \mathcal{H}}.$$ 

PROOF. Let $t \in T_{\cup \mathcal{H}}$. Then $\bigcup \{H_t : H \in \mathcal{H}\}$ is uncountable and is covered by the $\sigma$-closure-preserving closed family $\{H_t : H \in \mathcal{H}\}$. By Corollary 1 of Potoczny and Junnila [PJ], if each $H_t$ were compact, then $\bigcup \mathcal{H}$ would be the countable union of metacompact closed subsets of $[0,\omega_1)$. Since each metacompact closed subset of $[0,\omega_1)$ is countable, there is an $H \in \mathcal{H}$ such that $H_t$ is not compact. Hence $t \in T_H$.

From Claims 3 and 4 we get

CLAIM 5. If $\mathcal{H}$ is a $\sigma$-closure-preserving closed cover of $X_0$, then $\{T_H : H \in \mathcal{H}\}$ is a $\sigma$-closure-preserving closed cover of $T$.

Let $\mathcal{H}$ be a $\sigma$-closure-preserving closed cover of $X_0$. We shall show in Claim 7 that some element of $\mathcal{H}$ is not $C$-scattered.

CLAIM 6. $T_H$ is uncountable for some $H \in \mathcal{H}$.

PROOF. Since $T$ is hereditarily separable, the $\sigma$-closure-preserving cover $\{T_H : H \in \mathcal{H}\}$ of $T$ has a countable subcover. Hence the claim follows.

Let $H$ be an element of $\mathcal{H}$ such that $T_H$ is uncountable.

CLAIM 7. $H$ is not $C$-scattered.

PROOF. Let $\{t(n) : n < \omega\}$ be a countable self-dense subset of $T_H$. Each $H_t(n)$ is a closed unbounded set in $[0,\omega_1)$, hence there exists an $\alpha$ in $\bigcap \{H_t(n) : n < \omega\}$. Let $K$ be the closure of $\{(t(n), \alpha) : n < \omega\}$ in $X_0$. Then $K$ is a countable self-dense closed subset of $H$. Hence $H$ is not $C$-scattered.
REFERENCES


[N] T. Nogura, A compact-like space which does not have a countable cover by C-scattered closed subsets, Proc. Japan Acad. 59 (1983), 83-84.


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