A NOTE ON THE BORSUK-ULAM THEOREM

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ABSTRACT. Let $\mathcal{F}$ denote the set of all maps from $S^n$ to $\mathbb{R}^n$ topologized by the usual metric, and $\mathcal{B}$ the set of all nonempty closed subsets of $S^n$ invariant with respect to the antipodal map. Let $\beta: \mathcal{F} \to \mathcal{B}$ assign to each $f \in \mathcal{F}$ the set of all $x$ for which $f(x) = f(-x)$. The largest topology on $\mathcal{B}$ for which $\beta$ is continuous is identified: it is the upper semifinite topology.

Let $d$ denote the usual Pythagorean metric on $\mathbb{R}^n$ or $S^n$. Denote by $\mathcal{F}$ the set of all maps from $S^n$ to $\mathbb{R}^n$ and by $\mathcal{B}$ the set of all nonempty closed subsets of $S^n$ which are invariant with respect to the antipodal map of $S^n$. The Borsuk-Ulam theorem asserts that for each $f \in \mathcal{F}$ there is some point $x \in S^n$ for which $f(x) = f(-x)$; it is readily shown that the set of all such points is a member of $\mathcal{B}$.

This note considers the continuity of the function $\beta$.

Topologize $\mathcal{F}$ using the metric derived from the usual metric on $\mathbb{R}^n$ in the usual way, and $\mathcal{B}$ by the upper semifinite topology. The upper semifinite topology, defined by Michael in [2] on the collection of all nonempty closed subsets of a topological space, has as basis $\{V^* : V$ is an open subset of $S^n\}$, where $V^* = \{C \in \mathcal{B} : C \subseteq V\}$. This topology is very weak; it is not even $T_1$.

THEOREM. The function $\beta: \mathcal{F} \to \mathcal{B}$ is continuous. Moreover when $\mathcal{F}$ has the usual metric topology, the upper semifinite topology is the largest topology on $\mathcal{B}$ for which $\beta$ is continuous.

PROOF. Firstly it is shown that $\beta$ is continuous. Suppose $f \in \mathcal{F}$ and $V$ is an open subset of $S^n$ with $\beta(f) \subseteq V$. Let

$$\varepsilon = \min\{d(f(x), f(-x)) : x \in S^n - V\}.$$

Then $\varepsilon > 0$ and if $g \in \mathcal{F}$ is within $\varepsilon/2$ of $f$, then $\beta(g) \subseteq V$. Thus $\beta^{-1}(V^\#)$ is a neighborhood of $f$ in $\mathcal{F}$.

To show that the upper semifinite topology is the largest topology, let $\mathcal{U} \subseteq \mathcal{B}$ be such that $\beta^{-1}(\mathcal{U})$ is open. It will be shown that $\mathcal{U}$ is open in the upper semifinite topology, i.e.

$$\forall C \subseteq \mathcal{U}, \exists \text{ open } V \subseteq S^n \text{ with } C \subseteq V \text{ and } V^\# \subseteq \mathcal{U}.$$

Let $C \subseteq \mathcal{U}$.

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For each \( x \in S^n \), let \( \sigma_x : S^n \to \mathbb{R}^n \) be some stereographic projection from \(-x\) of the closed hemisphere centered at \( x \) onto \( B^n \) and if \( y \) is in the complementary hemisphere let \( \sigma_x(y) = -\sigma_x(-y) \). Then \( \sigma_x(y) = \sigma_x(-y) \) if and only if \( y = \pm x \).

For each \( x \in C \) define \( f_x : S^n \to \mathbb{R}^n \) by \( f_x(y) = d(y,C)\sigma_x(y) \). Then \( f_x \in \mathcal{F} \) and \( \beta(f_x) = C \) so \( f_x \in \beta^{-1}(\mathcal{U}) \). Thus \( \exists \varepsilon_x > 0 \) such that if \( g \in \mathcal{F} \) is within \( 2\varepsilon_x \) of \( f_x \), then \( g \in \beta^{-1}(\mathcal{U}) \). Let

\[
U_x = \{ y \in S^n : d(y,C) < \varepsilon_x \}
\]

and choose \( \delta_x \in (0, 1/2) \) so that \( B(x; 2\delta_x) \subset U_x \), where \( B(x;r) \) denotes the open ball in \( S^n \) of radius \( r \). Let \( \{B(x_i; \delta_x_i) : i = 1, \ldots, m\} \) be a finite subcover of the open cover \( \{B(x; \delta_x) : x \in C\} \) of \( C \).

Set

\[
V = \bigcap_{i=1}^m U_{x_i} \cap \left[ \bigcup_{i=1}^m B(x_i; \delta_x_i) \right].
\]

The set \( V \) is open and contains \( C \). Suppose \( D \in V^\# \). Since \( D \subset V \) and \( D \neq \emptyset \), there is an index \( i \) with \( D \cap B(x_i; \delta_x_i) \neq \emptyset \), say \( x \in D \cap B(x_i; \delta_x_i) \). Note also that \( D \subset U_{x_i} \). Define \( \varphi : S^n \to \mathbb{R} \) by

\[
\varphi(y) = \max\{d(y,C), \varepsilon_x\}
\]

and \( \psi : S^n \to \mathbb{R}^n \) by

\[
\psi(y) = \begin{cases} 
\sigma_{x_i}(y) & \text{if } y \in S^n - B(\pm x_i; 2\delta_x_i), \\
0 & \text{if } y = \pm x, \\
\frac{d(\sigma_{x_i}(y), \sigma_{x_i}(\pm x))}{d(\sigma_{x_i}(z), \sigma_{x_i}(\pm x))} \sigma_{x_i}(z) & \text{if } y \in B(\pm x_i; 2\delta_x_i) - \{\pm x\}.
\end{cases}
\]

In the last line, \( z \in \partial B(\pm x_i; 2\delta_x_i) \) is chosen so that \( \sigma_{x_i}(\pm x), \sigma_{x_i}(y), \) and \( \sigma_{x_i}(z) \) are collinear and in that order. It is readily checked that \( \psi \) is continuous and that \( \psi(y) = \psi(-y) \) if and only if \( y = \pm x \).

Define \( g : S^n \to \mathbb{R}^n \) by

\[
g(y) = \frac{d(y, D)}{d(y, D) + d(y, S^n - U_{x_i})} \varphi(y)\psi(y).
\]

Then \( g \in \mathcal{F} \) and is within \( 2\varepsilon_x \) of \( f_x \), so \( g \in \beta^{-1}(\mathcal{U}) \) and hence \( \beta(g) \in \mathcal{U} \). Since \( \beta(g) = D \) it follows that \( D \in \mathcal{U} \) so \( V^\# \subset \mathcal{U} \) as required. This completes the proof.

The theorem above may be compared with the results contained in [1], where cohomological bounds are obtained for sets derived from the Borsuk-Ulam sets of a parametrized family of maps.

It follows from the theorem that \( \beta \) is surjective, i.e. every nonempty closed subset of \( S^n \) which is invariant with respect to the antipodal map is the Borsuk-Ulam set of some map \( S^n \to \mathbb{R}^n \). This fact is comparable with Theorem 1 of [3].

REFERENCES


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