BEST CONSTANT FOR THE RATIO OF THE FIRST TWO EIGENVALUES OF ONE-DIMENSIONAL SCHÖDINGER OPERATORS WITH POSITIVE POTENTIALS
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ABSTRACT. We prove the optimal upper bound $\lambda_2/\lambda_1 \leq 4$ for the ratio of the first two eigenvalues of one-dimensional Schrödinger operators with nonnegative potentials. Equality holds if and only if the potential vanishes identically.

In this note we improve upon the one-dimensional portion of a theorem of Singer, Wong, Yau, and Yau [7] concerning the first two eigenvalues of a Schrödinger operator on a bounded domain with Dirichlet boundary conditions. Our result is best possible for the class of nonnegative potentials; in addition, it is an easy consequence of a commutation formula and the Rayleigh-Ritz inequality.

THEOREM. Let $[a, b]$ be a finite interval and let $V$ be a nonnegative real-valued function in $L^1(a, b)$. Let $H = -d^2/dx^2 + V(x)$ with Dirichlet boundary conditions act on $L^2(a, b)$, and let $\lambda_1[V]$ and $\lambda_2[V]$ denote the first two eigenvalues of $H$. Then $\lambda_2[V]/\lambda_1[V] \leq 4$ with equality if and only if $V = 0$.

PROOF. Let $u_1$ be the groundstate eigenfunction of $H$, i.e. $u_1$ is in the domain of $H$ and $Hu_1 = \lambda_1 u_1$. By the commutation formula (see, in particular, [3] and also [1, 2, 5]), the operator $\tilde{H} = -d^2/dx^2 + V - 2(u_1'/u_1)'$ has the same spectrum as $H$ except for $\lambda_1$. Thus we can obtain an upper bound for $\lambda_2$ by using the Rayleigh-Ritz inequality on $\tilde{H}$. Taking $u_2$ as our trial function we find $\tilde{H}u_2 = -4u_1'' + V u_2 - 3V u_2^2$ and hence

$$\lambda_2 \leq 4\lambda_1 - \frac{\int_a^b V u_2^2 \, dx}{\int_a^b u_2^2 \, dx}.$$ (1)

This shows that $\lambda_2/\lambda_1 \leq 4$ for nonnegative potentials. Moreover, equality holds if and only if $V = 0$ since $u_1$ is positive and continuous on $(a, b)$. □

REMARKS. 1. The choice of $u_2^2$ as the trial function was motivated by the fact that $u_2^2$ is the groundstate wavefunction of $\tilde{H}$ when $V = 0$.

2. $H$ above is to be interpreted in the sense of quadratic forms following [3], for example.

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3. Our result reads $\lambda_2 - \lambda_1 \leq 3\lambda_1$ when stated as a result about $\lambda_2 - \lambda_1$. This improves upon the bound $\lambda_2 - \lambda_1 \leq 4\lambda_1$ found in [7 and 1]. Such a bound (for general dimension) had also been obtained by Harrell [4] based on the work of Payne, Pólya, and Weinberger [6]. The paper [1] derived the bound $\lambda_2 - \lambda_1 \leq 4\lambda_1$ using the commutation formula and the Rayleigh-Ritz inequality in much the way we do above but with $u_1$ rather than $u_1^2$ as trial function.

4. We note that formula (1) holds for all $V$, not just for nonnegative $V$. In addition, we would expect formula (1) and the inequality $\lambda_2/\lambda_1 \leq 4$ to hold for singular potentials ($V \in L^1_{\text{loc}}(a,b)$, say) and also for problems on the line and half-line where $V(x) \to \infty$ as $x \to \pm \infty$. Of course, we always assume that Dirichlet boundary conditions are imposed at limit circle endpoints. Formally, all these results follow by commutation but one needs to take care that the operator $\tilde{H}$ generated in this way is well defined, selfadjoint, and has Dirichlet boundary conditions (when needed).

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NOTE ADDED IN PROOF. In connection with our remark 4, it has been pointed out to us by Barry Simon that extension of the bound $\lambda_2/\lambda_1 \leq 4$ to the line and half-line with $V(x) \to \infty$ as $|x| \to \infty$ follows from the given case by a limiting argument; in particular, the operator obtained by inserting infinite walls at $\pm a$ or $0$ and $a$ (infinite walls = Dirichlet boundary conditions) converges to the original operator in norm resolvent sense as $a \to \infty$ and this guarantees convergence of the eigenvalues. Since the bound holds for all finite-domain problems with Dirichlet boundary conditions it must continue to hold in the limit.

REFERENCES