UNCONDITIONAL BASES IN $L^2(0, a)$

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ABSTRACT. A method is given for producing unconditional bases in subspaces $K_\theta = H^2 \ominus \theta H^2$ of the Hardy space $H^2$, $\theta$ being an inner function in the upper half-plane. For $\theta = \exp(iaz)$ the space $K_\theta$ is the Fourier-Laplace transform of $L^2(0, a)$, which allows us to establish a necessary and sufficient condition for certain families of functions (including exponentials) to constitute unconditional bases in $L^2(0, a)$.

1. A family of nonzero vectors $\{e_n\}$ in a Hilbert space $H$ is called an unconditional basis in $H$ if every element $x$ in $H$ can be uniquely decomposed in an unconditionally convergent series

$$x = \sum_n \alpha_n e_n, \quad \alpha_n \in \mathbb{C}.$$ 

The Köthe-Toeplitz theorem says that a complete family $\{e_n\}$ is an unconditional basis iff for some $c$, $0 < c < 1$, the following "approximate Parseval identity" holds:

$$c \cdot \sum_n |\alpha_n|^2 \cdot \|e_n\|^2 \leq \left\| \sum_n \alpha_n e_n \right\|^2 \leq c^{-1} \sum_n |\lambda_n|^2 \cdot \|e_n\|^2$$

for every finite sequence $\{\alpha_n\}$ of complex numbers.

A classical example of an unconditional basis, in fact orthogonal, is given by the family of exponentials $\{e^{inx}\}_{n \in \mathbb{Z}}$, $H = L^2(0, 2\pi)$. Unconditional bases of exponentials $\{e^{i\lambda_n x}\}$ in $L^2(0, a)$ have been described in [1] under the assumption $\inf_n \text{Im} \lambda_n > -\infty$ ($\sup_n \text{Im} \lambda_n < +\infty$). The functions $w$ satisfying the Muckenhoupt condition (A2) on the real line $\mathbb{R}$,

$$\sup_{I \in \mathcal{I}} \left( \frac{1}{|I|} \int_I w \, dx \right) \cdot \left( \frac{1}{|I|} \int_I w^{-1} \, dx \right) < +\infty,$$

$I$ being the family of finite intervals, play an important role in this description. It was observed later [2] that the theory of Hankel operators permits one to extend the result of [1] to families of reproducing kernels of the Hardy class $H^2$ in the upper half-plane $\mathbb{C}_+$ (see [3] for details). The methods developed in [1 and 3] can be applied to the description of the resonance frequencies of semi-infinite strings corresponding to unconditional bases of resonance states [4]. One more application
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has been considered recently in [5] to investigate the unconditional basis property of \( \{x^{\beta-1}e_{1,\beta}(ix\lambda_n)\} \) in \( L^2(0,a) \). Here
\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}
\]
is a Mittag-Leffler function [6]. In this paper the results of [5] are extended to more general families. The advantage of the approach presented is that it makes the arguments more transparent even in the case of exponentials (\( \beta = 1 \)).

2. Our main tool is Widom's criterion for invertibility of Toeplitz operators \( T_\varphi \) with unimodular symbols \( \varphi \). We recall that the Toeplitz operator \( T_\varphi \) with symbol \( \varphi \), \( \varphi \) being a bounded measurable function on \( \mathbb{R} \), \( (\varphi \in L^\infty(\mathbb{R})) \) is defined by
\[
T_\varphi f = P_+(\varphi f), \quad f \in H^2.
\]
Here \( P_+ \) stands for the orthogonal projection in \( L^2(\mathbb{R}) \) onto \( H^2 \). We define the Hilbert transform \( \tilde{v} \) of \( v \in L^\infty(\mathbb{R}) \) as follows:
\[
\tilde{v}(x) = \frac{1}{\pi} \text{v.p.} \int \left\{ \frac{1}{x-t} + \frac{t}{1 + i t^2} \right\} v(t) \, dt.
\]
We state Widom's theorem in the form used in [3] (see Theorem 5). We say that a unimodular function \( \varphi \) is a Helson-Szegö function if \( T_\varphi \) is invertible. We denote by \( H^\infty \) the Hardy algebra in \( \mathbb{C}^+ \).

**THEOREM (WIDOM [7]).** Let \( \varphi \) be a unimodular function. Then the following are equivalent.
1. \( \varphi \) is a Helson-Szegö function.
2. \( \text{dist}_{L^\infty}(\varphi, H^\infty) < 1 \), \( \text{dist}(\overline{\varphi}, H^\infty) < 1 \).
3. There is an outer function \( f \in H^\infty \) such that \( \|\varphi - f\|_\infty < 1 \).
4. \( \varphi = \exp\{i(\overline{u} + v + c)\} \), where \( u, v \in L^\infty(\mathbb{R}) \), \( \|v\|_\infty < \pi/2 \), \( c \in \mathbb{R} \).
5. There are a constant \( \lambda \), \( |\lambda| = 1 \), and an outer function \( h \) with \( |h|^2 \) satisfying the Muckenhoupt condition \( (A_2) \) on \( \mathbb{R} \) such that
\[
\varphi = \lambda \overline{h}/h.
\]

**REMARK.** Notice that \( T_\varphi \) is left-invertible for a unimodular \( \varphi \) iff \( \text{dist}(\varphi, H^\infty) < 1 \) (see [3]).

Widom’s theorem, as stated above, provides a simple proof of the following well-known lemma. We put \( z = x + iy \), \( P_z = \pi^{-1}y \cdot (x^2 + y^2)^{-1} \) and denote by \( P_z^* f \) the harmonic extension of \( f \) from \( \mathbb{R} \) to \( \mathbb{C}^+ \).

**LEMMA 1.** Let \( h \) be an outer function in \( \mathbb{C}^+ \). Then \( |h|^2 \) satisfies (1) on \( \mathbb{R} \) if and only if
\[
P_z |h|^2 \leq \text{const} \cdot |h(z)|^2, \quad P_z |h|^{-2} \leq \text{const} \cdot |h(z)|^{-2}.
\]

**PROOF.** If \( |h|^2 \) satisfies (1) then \( T_{h/h} \) is invertible and 2 of Widon's theorem implies \( \text{dist}(\overline{h} \cdot h^{-1}, H^\infty) < 1 \). It follows that there exist \( f \in H^\infty \) and \( q < 1 \) such that
\[
||h(x)||^2 - f(x) \cdot h^2(x) < q|h(x)|^2, \quad x \in \mathbb{R}.
\]
Since \( P_z f h^2 = f(z) \cdot h^2(z) \) and \( ||f||_\infty < 2 \), we obtain
\[
P_z |h|^2 \leq q \cdot P_z |h|^2 + 2|h(z)|^2, \quad \text{i.e.} \quad P_z |h|^2 \leq 2(1-q)^{-1} \cdot |h(z)|^2.
\]
Similarly one can prove the second inequality. The converse is a trivial estimate of the Poisson kernel. \( \square \)

A sequence \( \{\lambda_n\} \) in \( \mathbb{C}_+ \) is said to be interpolating if for every bounded sequence \( \{\alpha_n\} \) there is \( f \in H^\infty \) such that \( f(\lambda_n) = \alpha_n \). The Carleson theorem [8] says that \( \{\lambda_n\} \) is interpolating iff
\[
\inf_n \prod_{k \neq n} \left| \frac{\lambda_k - \lambda_n}{\lambda_k - \lambda_n} \right| > 0. \tag{2}
\]
Clearly, (2) implies the Blaschke condition in \( \mathbb{C}_+ \) which allows one to consider the Blaschke product
\[
B = \prod_n \varepsilon_n \frac{z - \lambda_n}{z - \lambda_n},
\]
where the unimodular constants \( \varepsilon_n \) are chosen so as to provide the convergence of the product in \( \mathbb{C}_+ \).

Any inner function \( \theta \) in \( \mathbb{C}_+ \) generates a subspace \( K_\theta = H^2 \ominus \theta H^2 \) of \( H^2 \). We denote by \( P_\theta \) the orthogonal projection onto \( K_\theta \). Clearly \( P_\theta = \theta(I - P_+)\theta \).

**Theorem 1.** Let \( h \) be an outer function in \( \mathbb{C}_+ \) satisfying \( |h|^2 \in (1) \). \( \theta \) an inner function in \( \mathbb{C}_+ \), \( \{\lambda_n\} \) a sequence in \( \mathbb{C}_+ \) such that
\[
\sup_n |\theta(\lambda_n)| < 1. \tag{3}
\]
Then \( \{P_\theta(h(z - \lambda_n)^{-1})\} \) is an unconditional basis in \( K_\theta \) if and only if
(a) \( \{\lambda_n\} \) is an interpolating sequence; and
(b) \( h \cdot h^{-1} \cdot B \theta \) is a Helson-Szegö function, \( B \) being the Blaschke product associated with \( \{\lambda_n\} \).

**3. An application.** We observe first that \( h(z - \bar{\lambda}_n)^{-1} \in H^2 \) in view of Lemma 1. Now we show how to deduce the main result of [5] from this theorem. Let \( \mathcal{E}_\alpha \) be the set of entire functions with conjugate indicator diagram \([0, i\alpha], \beta \in \mathbb{R}, \{\lambda_n\} \) a sequence satisfying \( \inf_n \text{Im} \lambda_n > 0, B \) the Blaschke product associated with \( \{\lambda_n\}, \theta^\alpha(z) = \exp(i\alpha z) \).

**Theorem (Gubreev [5]).** For \( \beta \in (1/2, 3/2) \) the family \( \{x^{\beta-1} E_{1, \beta}(i x \lambda_n)\} \)
is an unconditional basis in \( L^2(0, a) \) if and only if
(a) \( \{\lambda_n\} \) is an interpolating sequence; and
(b) there is an entire function \( F \) in \( \mathcal{E}_\alpha \) with simple zeros \( \{\lambda_n\} \) such that
\[
|x|^{2(\beta-1)} |F(x)|^2 \text{ satisfies the Muckenhoupt condition.}
\]

**Proof.** This theorem is in fact a partial case of Theorem 1 with \( h = z^{1-\beta} \).
To see this we put \( x_+ = \max(x, 0) \) and note that \( (x_+^{\mu-1}/\Gamma(\mu)) \ast (x_+^{\mu-1}/\Gamma(\mu)) = x_+^{\mu+\mu-1}/\Gamma(\alpha + \mu) \) (see [9]). Hence, applying the operator of fractional integration
\[
I_\alpha f(x) \overset{\text{def}}{=} \frac{x_+^{\alpha-1}}{\Gamma(\alpha)} \ast f = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt
\]
with $\alpha = \beta - 1$ to the Taylor series of $\exp(i\lambda x)$, we obtain $I_{\rho-1}\exp(i\lambda x) = x^{\beta-1}E_{1,\beta}(ix\lambda)$. Next,

$$ \frac{1}{\Gamma(\beta)} \int_0^\infty x^{\beta-1}e^{i\beta x} \, dx = e^{i\beta\pi/2}z^{-\beta}, \quad z \in \mathbb{C}_+ $$

(see [9]), which yields the following formula for the Fourier-Laplace transform of $x^{\beta-1}E_{1,\beta}(ix\lambda)$, $\lambda \in \mathbb{C}_+$:

$$ \int_0^\infty x^{\beta-1}E_{1,\beta}(ix\lambda)e^{i\beta x} \, dx = e^{i\beta\pi/2}(\lambda + z)^{-1} \cdot z^{1-\beta}. $$

The Fourier-Laplace transform $f \to \hat{f}$ is an isometry of $L^2(0, +\infty)$ onto $H^2$ which maps $L^2(0, a)$ onto $K_\theta$. The function $z^{1-\beta}$ is outer in $\mathbb{C}_+$ and $|x|^{2(1-\beta)}$ satisfies (1) for $\beta \in \left(\frac{1}{2}, \frac{3}{2}\right)$. Therefore we can put $h = zx^{-\beta}$ in Theorem 1.

To finish the proof we need the following lemma. For an outer function $h$ satisfying $|h|^2 \in (1)$ we consider the class $M_{h,a}$ of entire functions $F \in \mathcal{E}_a$ such that $|\mathcal{E}_{a}^{-1}| \leq 1$.

**Lemma 2.** Let $\{\lambda_n\}$ be a Blaschke sequence in $\mathbb{C}_+$ with $\inf \text{Im} \lambda_n > 0$. Then the following are equivalent:

1. There is an $F \in M_{h,a}$ with simple zeros $\{\lambda_n\}$.
2. $h \cdot h^{-1}B_{\theta}a$ is a Helson-Szego function.

A proof of the lemma follows the same line as the proof of Theorem 1.2 (Part III of [3]), which deals with the case $h \equiv 1$. \qed

For $\beta \leq \frac{1}{2}$ the family $\{x^{\beta-1}E_{1,\beta}(ix\lambda)\}$ is not a basis in $L^2(0, a)$ since no element of the family belongs to $L^2(0, a)$. To see that the case $\beta \geq \frac{3}{2}$ is also impossible is more difficult and we refer the interested reader to [5].

**4. Proof of Theorem 1.** Let $N$ and $M$ be linear subspaces of a linear space $H$. We say that $H$ is a direct sum of $N$ and $M$ (notationally $H = N + M$) if $N \cap M = \{0\}$ and $H = N + M$.

We begin with two geometrical lemmas.

**Lemma 3.** Let $N$ and $M$ be closed subspaces of a Banach space $H$ and $T$ be a bounded operator from $H$ to a Banach space $X$ such that $\text{Ker} T = N$ and $TH = X$. Then $H = N + M$ if and only if $T$ maps $M$ isomorphically onto $X$.

**Proof.** If $H = N + M$ then $T : M \to X$ is an isomorphism by the Banach theorem. Let now $T$ map $M$ isomorphically onto $X$. Pick any $h \in H$ and consider $m = (T|M)^{-1}Th \in M$. Then obviously $n = h - m \in \text{Ker} T = N$, i.e. $h = n + m$. \qed

**Lemma 4.** The orthogonal projection $P_G$ onto a subspace $G$ of a Hilbert space $H$ maps a subspace $F$ isomorphically onto $G$ if and only if $H = F \perp G^\perp$, where $G^\perp = H \ominus G$.

**Proof.** Put $X = G$, $T = P_G$ and apply Lemma 3. \qed

We are going to apply Lemma 3 with $T$ being a Toeplitz operator. The following lemma justifies these applications.
LEMMA 5. Let \( \varphi \) be a Helson-Szegö function and \( I \) an arbitrary inner function. Then \( T_{\varphi I}(H^2) = H^2 \).

PROOF. The desired equality is an immediate consequence of the factorization \( T_{\varphi I} = T_I T_{\varphi} \).

LEMMA 6. Let \( h \) be an outer function and \( \{\lambda_n\} \) an arbitrary Blaschke sequence in \( C_+ \). Let \( |h|^2 \in (1) \) and \( \theta \) be an inner function. Then \( P_\theta \) is an isomorphism from the closed linear span of \( \{h(z - \lambda_n)^{-1}\} \) onto \( K_\theta \) if and only if \( \frac{h}{h^{-1}} \cdot B \cdot \theta \) is a Helson-Szegö function, \( B \) being the Blaschke product with zeros \( \{\lambda_n\} \).

PROOF. Let \( F \) be the closed linear span of \( \{h(z - \lambda_n)^{-1}\} \) and \( G = K_\theta \). By Lemma 4 the projection \( P_\theta \) maps \( F \) isomorphically onto \( G \) iff \( H^2 = F + G \). Put now \( N = F, M = G^\perp = \theta H^2, X = H = H^2, T = T_\frac{h}{h^{-1}}B \) in Lemma 3. Lemma 5 guarantees that \( T_{\frac{h}{h^{-1}}B}(H^2) = H^2 \). To check that \( \ker T_{\frac{h}{h^{-1}}B} = F \) we observe that \( T_{\frac{h}{h^{-1}}B} = T_B \cdot T_{\frac{h}{h^{-1}}} \). This yields \( \ker T_{\frac{h}{h^{-1}}B} = (T_{\frac{h}{h^{-1}}})^{-1}(\ker T_B) \). Now \( \ker T_B = K_B = \text{span}\{z - \lambda_n\}^{-1}\}. Besides, we have

\[
\frac{h}{h^{-1}}(z - \lambda_n)^{-1} = P_\frac{h}{h^{-1}}(z - \lambda_n)^{-1} = \frac{h(\lambda_n)}{h(\lambda_n)}(z - \lambda_n)^{-1},
\]

which implies the desired conclusion \( \ker T = F \), since \( T_{\frac{h}{h^{-1}}} \) is invertible.

Lemma 3 says that \( H^2 = F + G \) iff \( T_{\frac{h}{h^{-1}}B} : G^\perp \to H^2 \) is an isomorphism. The latter is, clearly, equivalent to the fact that \( (h/h) B \theta \) is a Helson-Szegö function. □

Let (a) and (b) of Theorem 1 hold. Then (a) implies that \( \{z - \lambda_n\}^{-1}\} \) is an unconditional basis in \( K_B \) [10] (see also [3]). Since \( T_{\frac{h}{h^{-1}}} \) is invertible, it follows from (4) that \( \{h(z - \lambda_n)^{-1}\} \) is an unconditional basis in its span \( F \). Now (b) and Lemma 6 imply that \( P_\theta \) maps \( F \) isomorphically onto \( K_\theta \), which proves the theorem in one direction.

Let \( \{P_\theta(h(z - \lambda_n)^{-1})\} \) be an unconditional basis in \( K_\theta \). We recall that a family \( \{e_n\} \) of vectors of a Banach space is called uniformly minimal if there is a biorthogonal family \( \{f_n\} \) of bounded functionals \( \langle e_n, f_m \rangle = \delta_{nm} \) such that \( \sup_n \|e_n\| \cdot \|f_n\| < \infty \). If \( \{e_n\} \) is an unconditional basis then it is uniformly minimal. It is clear that if \( T_{e_n^*} = e_n \) and \( \|e_n\| \geq c \cdot ||e_n^*|| \) then \( \{e_n^*\} \) is also uniformly minimal. Set \( f_n^* = T^* f_n \). Then \( \|e_n^*\| \cdot \|f_n^*\| < c^{-1} \cdot \|T^*\| \cdot ||e_n|| \cdot ||f_n|| \).

LEMMA 7. If \( \sup_n ||\theta(\lambda_n)|| < 1 \) then there is a positive constant \( c \) such that

\[
\|P_\theta(h(z - \lambda_n)^{-1})\| \geq c \|h(z - \lambda_n)^{-1}\|.
\]

PROOF. We have

\[
\|P_\theta(h(z - \lambda_n)^{-1})\| = \|(I - P_+) (h \theta)(x - \lambda_n)^{-1})\|_{L^2} = \|P_+ \theta h(x - \lambda_n)^{-1}\|_{L^2},
\]

\[
= \left| \frac{\frac{h}{h^{-1}} \cdot \theta(x) - \theta(\lambda_n)}{x - \lambda_n} \right| \left( \|T_{\frac{h}{h^{-1}}B}(x)\|^{-1} \cdot \|\theta(x) - \theta(\lambda_n)\| \right),
\]

\[
\geq \left( 1 - \sup_n ||\theta(\lambda_n)|| \right) \left( \|T_{\frac{h}{h^{-1}}B}(x)\|^{-1} \cdot ||h(x - \lambda_n)^{-1}\||_{H^2} \right).
\]

Since \( T_{\frac{h}{h^{-1}}} \) is invertible, (4) implies that \( \{(z - \lambda_n)^{-1}\} \) is uniformly minimal too. It is well known that \( \{B(z - \lambda_n)^{-1}(iB'(\lambda_n))^{-1}\} \) is a biorthogonal family for
\{(z - \lambda_n)^{-1}\}, which yields \(\inf_n |B'(\lambda_n)|/\text{Im } \lambda_n > 0\) and therefore \{\lambda_n\} is interpolating. Then \{(z - \lambda_n)^{-1}\} is an unconditional basis in \(K_B\). From (4) we obtain that \(\{h(z - \lambda_n)^{-1}\}\) is an unconditional basis in \(F\) because \(T_{h^{-1}\theta}^{\theta}\) is invertible. Lemma 7 says that \(\rho_\theta\) does not distort the elements \(h(z - \lambda_n)^{-1}\). This clearly implies that \(\rho_\theta : F \rightarrow K_\theta\) is an isomorphism. Lemma 6 guarantees that \(\bar{h} \cdot h^{-1} \cdot \bar{\theta}\) is a Helson-Szegö function. □

REFERENCES


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