UNCONDITIONAL BASES IN $L^2(0, a)$

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ABSTRACT. A method is given for producing unconditional bases in subspaces $K_\theta = H^2 \ominus \theta H^2$ of the Hardy space $H^2$, $\theta$ being an inner function in the upper half-plane. For $\theta = \exp(iaz)$ the space $K_\theta$ is the Fourier-Laplace transform of $L^2(0, a)$, which allows us to establish a necessary and sufficient condition for certain families of functions (including exponentials) to constitute unconditional bases in $L^2(0, a)$.

1. A family of nonzero vectors \( \{e_n\} \) in a Hilbert space $H$ is called an unconditional basis in $H$ if every element $x$ in $H$ can be uniquely decomposed in an unconditionally convergent series

\[
x = \sum_n \alpha_n e_n, \quad \alpha_n \in \mathbb{C}.
\]

The Köthe-Toeplitz theorem says that a complete family $\{e_n\}$ is an unconditional basis iff for some $c$, $0 < c < 1$, the following "approximate Parseval identity" holds:

\[
c \cdot \sum_n |\alpha_n|^2 \cdot \|e_n\|^2 \leq \left\| \sum_n \alpha_n e_n \right\|^2 \leq c^{-1} \sum_n |\lambda_n|^2 \cdot \|e_n\|^2
\]

for every finite sequence $\{\alpha_n\}$ of complex numbers.

A classical example of an unconditional basis, in fact orthogonal, is given by the family of exponentials $\{e^{inx}\}_{n \in \mathbb{Z}}$, $H = L^2(0, 2\pi)$. Unconditional bases of exponentials $\{e^{inx}\}$ in $L^2(0, a)$ have been described in [1] under the assumption $\inf_n \text{Im} \lambda_n > -\infty$ ($\sup_n \text{Im} \lambda_n < +\infty$). The functions $w$ satisfying the Muckenhoupt condition (A2) on the real line $\mathbb{R}$,

\[
(1) \quad \sup_{I \in \mathcal{I}} \left( \frac{1}{|I|} \int_I w \, dx \right) \cdot \left( \frac{1}{|I|} \int_I w^{-1} \, dx \right) < +\infty,
\]

$I$ being the family of finite intervals, play an important role in this description. It was observed later [2] that the theory of Hankel operators permits one to extend the result of [1] to families of reproducing kernels of the Hardy class $H^2$ in the upper half-plane $\mathbb{C}_+$ (see [3] for details). The methods developed in [1 and 3] can be applied to the description of the resonance frequencies of semi-infinite strings corresponding to unconditional bases of resonance states [4]. One more application...
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has been considered recently in [5] to investigate the unconditional basis property of \( \{ x^{\beta-1} E_{1,\beta}(ix\lambda n) \} \) in \( L^2(0,\alpha) \). Here

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}
\]

is a Mittag-Leffler function [6]. In this paper the results of [5] are extended to more general families. The advantage of the approach presented is that it makes the arguments more transparent even in the case of exponentials (\( \beta = 1 \)).

2. Our main tool is Widom’s criterion for invertibility of Toeplitz operators \( T_\varphi \) with unimodular symbols \( \varphi \). We recall that the Toeplitz operator \( T_\varphi \) with symbol \( \varphi \), \( \varphi \) being a bounded measurable function on \( \mathbb{R} \), \( (\varphi \in L^\infty(\mathbb{R})) \) is defined by

\[
T_\varphi f = P_+(\varphi f), \quad f \in H^2.
\]

Here \( P_+ \) stands for the orthogonal projection in \( L^2(\mathbb{R}) \) onto \( H^2 \). We define the Hilbert transform \( \tilde{v} \) of \( v \in L^\infty(\mathbb{R}) \) as follows:

\[
\tilde{v}(x) = \frac{1}{\pi} (\text{v.p.}) \int \left\{ \frac{1}{x-t} + \frac{t}{1+i^2} \right\} v(t) \, dt.
\]

We state Widom’s theorem in the form used in [3] (see Theorem 5). We say that a unimodular function \( \varphi \) is a Helson-Szegö function if \( T_\varphi \) is invertible. We denote by \( H^\infty \) the Hardy algebra in \( C_+ \).

**THEOREM (WIDOM [7]).** Let \( \varphi \) be a unimodular function. Then the following are equivalent.

1. \( \varphi \) is a Helson-Szegö function.
2. \( \text{dist}_{L^\infty}(\varphi, H^\infty) < 1, \text{dist}(\varphi, H^\infty) < 1. \)
3. There is an outer function \( f \in H^\infty \) such that \( \| \varphi - f \|_\infty < 1 \).
4. \( \varphi = \exp\{i(\tilde{u} + v + c)\} \), where \( u, v \in L^\infty(\mathbb{R}) \), \( \| v \|_\infty < \pi/2 \), \( c \in \mathbb{R} \).
5. There are a constant \( \lambda, |\lambda| = 1 \), and an outer function \( h \) with \( |h|^2 \) satisfying the Muckenhoupt condition \( (A_2) \) on \( \mathbb{R} \) such that

\[
\varphi = \lambda \frac{h}{h}.
\]

**REMARK.** Notice that \( T_\varphi \) is left-invertible for a unimodular \( \varphi \) iff \( \text{dist}(\varphi, H^\infty) < 1 \) (see [3]).

Widom’s theorem, as stated above, provides a simple proof of the following well-known lemma. We put \( z = x + iy, P_z = \pi^{-1} y \cdot (x^2 + y^2)^{-1} \) and denote by \( P_z f \) the harmonic extension of \( f \) from \( \mathbb{R} \) to \( C_+ \).

**LEMMA 1.** Let \( h \) be an outer function in \( C_+ \). Then \( |h|^2 \) satisfies (1) on \( \mathbb{R} \) if and only if

\[
P_z |h|^2 \leq \text{const} \cdot |h(z)|^2, \quad P_z |h|^{-2} \leq \text{const} \cdot |h(z)|^{-2}.
\]

**PROOF.** If \( |h|^2 \) satisfies (1) then \( T_{h^*}^{-1} \) is invertible and 2 of Widom’s theorem implies \( \text{dist}(\tilde{h} \cdot h^{-1}, H^\infty) < 1 \). It follows that there exist \( f \in H^\infty \) and \( q < 1 \) such that

\[
| |h(x)|^2 - f(x) \cdot h^2(x)| | < q |h(x)|^2, \quad x \in \mathbb{R}.
\]
Since $P_z f h^2 = f(z) \cdot h^2(z)$ and $||f||_{\infty} < 2$, we obtain

$$P_z |h|^2 \leq q \cdot P_z |h|^2 + 2|h(z)|^2,$$

i.e. $P_z |h|^2 \leq 2(1-q)^{-1} \cdot |h(z)|^2$.

Similarly one can prove the second inequality. The converse is a trivial estimate of the Poisson kernel. □

A sequence $\{\lambda_n\}$ in $\mathbb{C}_+$ is said to be interpolating if for every bounded sequence $\{\alpha_n\}$ there is $f \in H^\infty$ such that $f(\lambda_n) = \alpha_n$. The Carleson theorem [8] says that $\{\lambda_n\}$ is interpolating iff

$$\inf \prod_{n \neq m} \left| \frac{\lambda_k - \lambda_n}{\lambda_k - \lambda_m} \right| > 0.$$  \hspace{1cm} (2)

Clearly, (2) implies the Blaschke condition in $\mathbb{C}_+$ which allows one to consider the Blaschke product

$$B = \prod_n \varepsilon_n \frac{z - \lambda_n}{z - \lambda_n},$$

where the unimodular constants $\varepsilon_n$ are chosen so as to provide the convergence of the product in $\mathbb{C}_+$.

Any inner function $\theta$ in $\mathbb{C}_+$ generates a subspace $K_\theta = H^2 \ominus \theta H^2$ of $H^2$. We denote by $P_\theta$ the orthogonal projection onto $K_\theta$. Clearly $P_\theta = \theta(1 - P_+\theta)$.

**THEOREM 1.** Let $h$ be an outer function in $\mathbb{C}_+$ satisfying $|h|^2 \in (1)$. $\theta$ an inner function in $\mathbb{C}_+$, $\{\lambda_n\}$ a sequence in $\mathbb{C}_+$ such that

$$\sup_n |\theta(\lambda_n)| < 1.$$  \hspace{1cm} (3)

Then $\{P_\theta(h(z - \lambda_n)^{-1})\}$ is an unconditional basis in $K_\theta$ if and only if

(a) $\{\lambda_n\}$ is an interpolating sequence; and

(b) $\hat{h} \cdot h^{-1} \cdot \hat{B} \theta$ is a Helson-Szegö function, $B$ being the Blaschke product associated with $\{\lambda_n\}$.

**3. An application.** We observe first that $h(z - \lambda_n)^{-1} \in H^2$ in view of Lemma 1. Now we show how to deduce the main result of [5] from this theorem. Let $\mathcal{E}_\beta$ be the set of entire functions with conjugate indicator diagram $[0, i\beta]$, $\beta \in \mathbb{R}$, $\{\lambda_n\}$ a sequence satisfying $\inf_n \text{Im} \lambda_n > 0$, $B$ the Blaschke product associated with $\{\lambda_n\}$, $\theta^\alpha(z) = \exp(i\alpha z)$.

**THEOREM (GUBREEV [5]).** For $\beta \in (1/2, 3/2)$ the family $\{x^{\beta-1} E_1, \theta(i\alpha \lambda_n)\}$ is an unconditional basis in $L^2(0, a)$ if and only if

(a) $\{\lambda_n\}$ is an interpolating sequence; and

(b) there is an entire function $F$ in $\mathcal{E}_\beta$ with simple zeros $\{\lambda_n\}$ such that $|x|^{2(\beta-1)} |F(x)|^2$ satisfies the Muckenhoupt condition.

**PROOF.** This theorem is in fact a partial case of Theorem 1 with $h = z^{1-\beta}$. To see this we put $x_+^{\text{def}} = \max(x, 0)$ and note that $(x_+^{\alpha - 1}/\Gamma(\alpha)) * (x_+^{\mu - 1}/\Gamma(\mu)) = x_+^{\alpha + \mu - 1}/\Gamma(\alpha + \mu)$ (see [9]). Hence, applying the operator of fractional integration

$$I_\alpha f(x) \overset{\text{def}}{=} \frac{x_+^{\alpha - 1}}{\Gamma(\alpha)} * f = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t) \, dt$$
with $\alpha = \beta - 1$ to the Taylor series of $\exp(i\lambda x)$, we obtain $I_{\beta-1}\exp(i\lambda x) = x^{\beta-1}E_{1,\beta}(ix\lambda)$. Next,

$$\frac{1}{\Gamma(\beta)} \int_{0}^{\infty} x^{\beta-1} e^{ixx} \, dx = e^{i\beta\pi/2}z^{-\beta}, \quad z \in \mathbb{C}^+$$

(see [9]), which yields the following formula for the Fourier-Laplace transform of $x^{\beta-1}E_{1,\beta}(ix\lambda)$, $\lambda \in \mathbb{C}^+$:

$$\int_{0}^{\infty} x^{\beta-1}E_{1,\beta}(ix\lambda)e^{ixx} \, dx = e^{i\beta\pi/2}(\lambda + z)^{-1} \cdot z^{1-\beta}.$$  

The Fourier-Laplace transform $f \rightarrow \hat{f}$ is an isometry of $L^2(0, +\infty)$ onto $H^2$ which maps $L^2(0, a)$ onto $K_{\theta^a}$. The function $z^{1-\beta}$ is outer in $\mathbb{C}^+$ and $|x|^{2(1-\beta)}$ satisfies (1) for $\beta \in (\frac{1}{2}, \frac{3}{2})$. Therefore we can put $h = z^{1-\beta}$ in Theorem 1.

To finish the proof we need the following lemma. For an outer function $h$ satisfying $|h|^2 \in (1)$ we consider the class $M_{h,a}$ of entire functions $F \in \mathcal{E}_a$ such that

$$|h \cdot h^{-1}B^a| \in (1).$$

**LEMMA 2.** Let $\{\lambda_n\}$ be a Blaschke sequence in $\mathbb{C}^+$ with $\inf \text{Im} \lambda_n > 0$. Then the following are equivalent:

1. There is an $F$ in $M_{h,a}$ with simple zeros $\{\lambda_n\}$.
2. $h \cdot h^{-1}B^a$ is a Helson-Szegö function.

A proof of the lemma follows the same line as the proof of Theorem 1.2 (Part III of [3]), which deals with the case $h \equiv 1$. □

For $\beta \leq \frac{1}{2}$ the family $\{x^{\beta-1}E_{1,\beta}(ix\lambda)\}$ is not a basis in $L^2(0, a)$ since no element of the family belongs to $L^2(0, a)$. To see that the case $\beta \geq \frac{3}{2}$ is also impossible is more difficult and we refer the interested reader to [5].

4. **Proof of Theorem 1.** Let $N$ and $M$ be linear subspaces of a linear space $H$. We say that $H$ is a direct sum of $N$ and $M$ (notationally $H = N + M$) if $N \cap M = \{0\}$ and $H = N + M \overset{\text{def}}{=} \{n + m : n \in N, m \in M\}$.

We begin with two geometrical lemmas.

**LEMMA 3.** Let $N$ and $M$ be closed subspaces of a Banach space $H$ and $T$ be a bounded operator from $H$ to a Banach space $X$ such that $\text{Ker} \, T = N$ and $TH = X$. Then $H = N + M$ if and only if $T$ maps $M$ isomorphically onto $X$.

**PROOF.** If $H = N + M$ then $T: M \rightarrow X$ is an isomorphism by the Banach theorem. Let now $T$ map $M$ isomorphically onto $X$. Pick any $h \in H$ and consider $m = (T|M)^{-1}Th \in M$. Then obviously $n = h - m \in \text{Ker} \, T = N$, i.e. $h = n + m$. □

**LEMMA 4.** The orthogonal projection $P_G$ onto a subspace $G$ of a Hilbert space $H$ maps a subspace $F$ isomorphically onto $G$ if and only if $H = F \perp G^\perp$, where $G^\perp = H \cap G$.

**PROOF.** Put $X = G$, $T = P_G$ and apply Lemma 3. □

We are going to apply Lemma 3 with $T$ being a Toeplitz operator. The following lemma justifies these applications.
**Lemma 5.** Let $\varphi$ be a Helson-Szegö function and $I$ an arbitrary inner function. Then $T_{\varphi I}(H^2) = H^2$.

**Proof.** The desired equality is an immediate consequence of the factorization $T_{\varphi I} = T_I T_{\varphi}$.

**Lemma 6.** Let $h$ be an outer function and $\{\lambda_n\}$ an arbitrary Blaschke sequence in $\mathbb{C}^*$. Let $|h|^2 \in (1)$ and $\theta$ be an inner function. Then $P_\theta$ is an isomorphism from the closed linear span of $\{h(z - \bar{\lambda}_n)^{-1}\}$ onto $K_\theta$ if and only if $\bar{h} \cdot \theta^{-1} \cdot B \cdot \theta$ is a Helson-Szegö function, $B$ being the Blaschke product with zeros $\{\lambda_n\}$.

**Proof.** Let $F$ be the closed linear span of $\{h(z - \bar{\lambda}_n)^{-1}\}$ and $G = K_\theta$. By Lemma 4 the projection $P_\theta$ maps $F$ isomorphically onto $G$ iff $H^2 = F + G^\perp$. Put now $N = F$, $M = G^\perp = \theta H^2$, $X = H = H^2$, $T = T(\bar{h}/h)B$ in Lemma 3. Lemma 5 guarantees that $T(\bar{h}/h)B(H^2) = H^2$. To check that $\ker T(\bar{h}/h)B = F$ we observe that $T(\bar{h}/h)B = T_B \cdot T(h/h)$. This yields $\ker T(\bar{h}/h)B = (T(h/h)^{-1}(\ker T_B))$. Now $\ker T_B = K_B = \text{span}\{(z - \bar{\lambda}_n)^{-1}\}$. Besides, we have

$$T(\bar{h}/h)(h(z - \bar{\lambda}_n)^{-1}) = \frac{P_+(h(z - \bar{\lambda}_n)^{-1})}{h(\lambda_n)}(z - \bar{\lambda}_n)^{-1},$$

which implies the desired conclusion $\ker T = F$, since $T(\bar{h}/h)$ is invertible.

Lemma 3 says that $H^2 = F + G^\perp$ iff $T(\bar{h}/h)B : G^\perp \to H^2$ is an isomorphism. The latter is, clearly, equivalent to the fact that $(\bar{h}/h)B \theta$ is a Helson-Szegö function.

Let (a) and (b) of Theorem 1 hold. Then (a) implies that $\{(z - \bar{\lambda}_n)^{-1}\}$ is an unconditional basis in $K_B$ [10] (see also [3]). Since $T(\bar{h}/h)$ is invertible, it follows from (4) that $\{h(z - \bar{\lambda}_n)^{-1}\}$ is an unconditional basis in its span $F$. Now (b) and Lemma 6 imply that $P_\theta$ maps $F$ isomorphically onto $K_\theta$, which proves the theorem in one direction.

Let $\{P_\theta(h(z - \bar{\lambda}_n)^{-1})\}$ be an unconditional basis in $K_\theta$. We recall that a family $\{e_n\}$ of vectors of a Banach space is called uniformly minimal if there is a biorthogonal family $\{f_n\}$ of bounded functionals $(\langle e_n, f_m \rangle = \delta_{nm})$ such that $\sup_n \|e_n\| \cdot \|f_n\| < +\infty$. If $\{e_n\}$ is an unconditional basis then it is uniformly minimal. It is clear that if $T e_n^* = e_n$ and $\|e_n\| \geq c \cdot \|e_n^*\|$ then $\{e_n^*\}$ is also uniformly minimal. Set $f_n^* = T^* f_n$. Then $\|e_n^*\| \cdot \|f_n^*\| \leq c^{-1} \cdot \|T^*\| \cdot \|e_n\| \cdot \|f_n\|$. 

**Lemma 7.** If $\sup_n |\theta(\lambda_n)| < 1$ then there is a positive constant $c$ such that

$$\|P_\theta(h(z - \bar{\lambda}_n)^{-1})\| \geq c \|h(z - \bar{\lambda}_n)^{-1}\|.$$

**Proof.** We have

$$\|P_\theta(h(z - \bar{\lambda}_n)^{-1})\| = \|(I + P_+)(h\bar{\theta}(x - \bar{\lambda}_n)^{-1})\|_{L^2} = \|P_+ \theta \bar{h}(x - \lambda_n)^{-1}\|_{L^2},$$

$$= \|P_+ \bar{h} \cdot \frac{\theta(x) - \theta(\lambda_n)}{x - \lambda_n} \cdot h\|_{L^2} \geq (\|T_{\bar{h}/h}^{-1}\|^{-1} \cdot \|\frac{\theta(x) - \theta(\lambda_n)}{x - \lambda_n}\|_{L^2}),$$

$$\geq (1 - \sup_n |\theta(\lambda_n)|) \cdot (\|T_{\bar{h}/h}^{-1}\|^{-1} \cdot \|h(x - \bar{\lambda}_n)^{-1}\|_{L^2}).$$

Since $T_{\bar{h}/h}$ is invertible, (4) implies that $\{(z - \bar{\lambda}_n)^{-1}\}$ is uniformly minimal too. It is well known that $\{B(z - \lambda_n)^{-1}(IB'(\lambda_n))^{-1}\}$ is a biorthogonal family for
\{(z - \lambda_n)^{-1}\}, which yields \(\inf_n |B'(\lambda_n)|/\text{Im} \lambda_n > 0\) and therefore \(\{\lambda_n\}\) is interpolating. Then \(\{(z - \lambda_n)^{-1}\}\) is an unconditional basis in \(K_B\). From (4) we obtain that \(\{h(z - \lambda_n)^{-1}\}\) is an unconditional basis in \(F\) because \(T_{h/h}^k\) is invertible. Lemma 7 says that \(\mathcal{P}_\theta\) does not distort the elements \(h(z - \lambda_n)^{-1}\). This clearly implies that \(\mathcal{P}_\theta: F \rightarrow K_\theta\) is an isomorphism. Lemma 6 guarantees that \(\tilde{h} \cdot h^{-1} \cdot B\theta\) is a Helson-Szegö function. □

References


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