NOTE ON COMPATIBLE VECTOR TOPOLOGIES

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ABSTRACT. Let \( \langle X, Y \rangle \) be a dual pair. Then \( X \) admits the finest locally convex topology \( \mu \) which is compatible with \( \langle X, Y \rangle \). In contrast, it is proved that there is no finest vector topology on \( X \) which is compatible with \( \langle X, Y \rangle \) provided \( X \) contains a \( \mu \)-dense subspace of infinite codimension.

Introduction. Let \( \tau \) be a vector topology on a (vector) space \( X \) different from the finest one of \( X \) and compatible with a dual pair \( \langle X, Y \rangle \), i.e. \( Y \) is the topological dual of \( (X, \tau) \).

(a) Does there exist on \( X \) a vector topology which is strictly finer than \( \tau \) and compatible with \( \langle X, Y \rangle \)?

(b) Does there exist on \( X \) the finest vector topology compatible with \( \langle X, Y \rangle \)?

We prove that (a) has a positive and (b) a negative solution whenever \( X \) contains a \( \tau \)-dense subspace of infinite codimension. In fact we obtain a stronger result (Theorem 2). Some applications are also included.

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Notation. We consider only infinite dimensional Hausdorff topological vector spaces (tvs) \( X = (X, \tau) \). If \( G \) is a (vector) subspace of \( X \), then \( \tau \mid G \) and \( \tau/G \) denote the topology \( \tau \) restricted to \( G \) and the quotient topology of the quotient space \( X/G \), respectively. If \( \lambda \) is a finer vector topology on \( X/G \), we denote by \( \vartheta := \tau \vee \lambda \) the weakest vector topology on \( X/G \) such that \( \tau \leq \vartheta, \vartheta/G = \lambda \). By \( \text{sup}(\tau, \vartheta) \) [\( \text{inf}(\tau, \vartheta) \)] we denote the weakest [finest] vector topology on \( X \) which is finer [weaker] than \( \tau \) and \( \vartheta \). A tvs \( X \) (and its topology) will be called dual-less if \( X' = 0 \); \( X' \) and \( X^* \) denote the topological and algebraic dual of \( X \), respectively. A tvs \( X \) is semibornological [Mazur] if every bounded [sequentially continuous] linear functional on \( X \) is continuous.

Results. We shall need the next lemma; its proof combines some ideas found in [7 and 5].

Lemma 1. Every infinite dimensional vector space \( X \) admits a locally bounded dual-less topology.

Proof. Let \( \Gamma \) be a Hamel basis of \( X \). Let \( \vartheta \) be a vector topology defined by the norm \( \|x\| := \sum |t_s| \), where \( x = \sum t_s x_s, x_s \in \Gamma \). Clearly \( (X, \vartheta) \) is isomorphic to a dense subspace of the space \( l^1(\Gamma) \). Hence it is enough to find on \( l^1(\Gamma) \) a weaker...
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locally bounded dual-less topology. Fix $0 < p < 1$. Choose in the dual-less space $L^p := (L^p[0,1], \| \cdot \|_p)$ a sequence $(y_n)$ with the properties: $\sum \| y_n \|_p < \infty$; $\text{lin}(y_n)$ is dense in $L^p$; and $(y_n)$ is $m$-independent, i.e. if $(t_n) \in l^\infty$ and $\sum t_n y_n = 0$, then $(t_n) = 0$ [5, Theorem 1]. If $T(x) := \sum x_n y_n$, $x = (x_n) \in l^1$, then $T$ is a continuous injective linear map with dense range. $T$ induces a continuous injective linear map $T: l^1(\Gamma, l^1) \to l^1(\Gamma, L^p)$ also with dense range, where $l^1(\Gamma, L^p)$ is the locally bounded dual-less space of all functions $f = (f_s)$, $f_s \in L^p$, with $\sum \| f_s \|_p < \infty$. Since $l^1(\Gamma)$ and $l^1(\Gamma, l^1)$ are isomorphic, $l^1(\Gamma)$ admits a topology as claimed.

**Theorem 2.** Let $G$ be a dense infinite codimensional subspace of a tvs $(X, \tau)$. Then $X$ admits a strictly finer vector topology $\vartheta$ such that $\tau | G = \vartheta | G$, $\vartheta / G$ is locally bounded and dual-less, and $(X, \tau)' = (X, \vartheta)'$. Moreover, there is no finest vector topology $\alpha$ on $X$ such that $(X, \tau)' = (X, \alpha)'$.

**Proof.** Set $\vartheta := \tau \vee \varphi$, where $\varphi$ is a vector topology on $X / G$ as in Lemma 1. To finish the proof it is enough to find on $X$ strictly finer vector topologies producing the same continuous linear functionals as $\tau$, but whose supremum topology does not have this property. In view of Theorem B of [7] the finest vector topology $\gamma$ on $X / G$ is the supremum of dual-less topologies $\gamma^1, \gamma^2, \gamma^3$. Set $\vartheta^i := \tau \vee \gamma^i$ and $\vartheta := \sup(\vartheta^i : 1 \leq i \leq 3)$. Clearly $(X, \vartheta^i)' = (X, \tau)'$, $i = 1, 2, 3$. Moreover, $\vartheta / G = \gamma$. Then $(X, \tau)' \neq (X, \vartheta)'$ (because every nontrivial linear functional on $X$ which vanishes on $G$ is $\vartheta$-continuous but discontinuous for $\tau$).

Note that there exist tvs $X$ not carrying the finest vector topology of $X$ but which admit the finest vector topology compatible with $(X, X')$; every uncountable dimensional vector space $X$ with the weak topology $\sigma(X, X^*)$ provides an example of such a space.

**Remark 3.** Note that a tvs $(X, \tau)$ has a dense subspace of infinite codimension if $X$ contains an infinite dimensional subspace admitting a finer metrizable vector topology, compare [6, Theorem 1]. In particular, every boundedly summing tvs [1, p. 74] containing an infinite dimensional bounded subset has a dense subspace of infinite codimension (this generalizes Proposition 1.5 of [9]). Note that every bornological (or boundedly summing ultrabornological [4]) space $X$ with $X' \neq X^*$ enjoys this property, compare [1, (5), p. 76]. In particular, every semibornological lcs $X$ with $X' \neq X^*$ also has this property (because $X$ is bornological under the finest locally convex topology $\tau^b$ which produces the same bounded sets as $\tau$; clearly then $(X, \tau)' = (X, \tau^b)'$).

A vector topology $\tau$ on $X$ is said to have the Hahn Banach Extension Property (HBEP) if $(X, \tau)'$ separates points from the closed subspaces of $X$.

**Proposition 4.** Let $X$ be an lcs such that $(X, \mu)$ contains a dense bornological subspace of infinite codimension. Then there exist on $X$ two vector topologies $\vartheta^1$ and $\vartheta^2$ without the HBEP, compatible with $(X, X')$, such that $\mu = \inf(\vartheta^1, \vartheta^2)$. Moreover, $\vartheta^1$ and $\vartheta^2$ can be chosen to be metrizable [and ultrabarrelled] provided $\mu$ is metrizable [and complete].

**Proof.** Let $\vartheta^1$ be a vector topology on $X$ as in Theorem 2. Using Remark 3 we deduce that $(X, \vartheta^1)$ has a dense subspace $G$ of infinite codimension. By Theorem 2 we find on $X$ a vector topology $\vartheta^2$ which is compatible with $(X, X')$ and not compatible with $\vartheta^1$. Set $\gamma := \inf(\vartheta^1, \vartheta^2)$. Since $G$ is $\gamma$-dense, $\gamma$ is locally
convex. Hence $\mu = \gamma$. Applying Theorem 2.6 of [8] we obtain the last assertion of Proposition 4.

Applying Corollary 1.3 of [10] and our Remark 3 we deduce that every sequentially complete bornological space $X$ with $X' \neq X^*$ satisfies also the assumption of Proposition 4.

By the three space problem (for property $P$) we understand the following: Suppose $X$ is a tvs and $G$ is a subspace of $X$ such that $G$ and $X/G$ have property $P$. Does $X$ have property $P$? (see for example [8]).

The following fact (being an immediate consequence of the proof of Theorem 2) will be used to establish that the three space problem has a negative solution if $P$ is either semibornological or Mazur.

**Proposition 5.** Let $P$ and $P^0$ be certain properties of tvs such that: (1) Every metrizable tvs has property $P$. (2) If $(X, \tau)$ has property $P$, then $(X, \tau^0)$ has property $P^0$, where $\tau^0$ denotes the finest locally convex topology on $X$ weaker than $\tau$. If there exists a Mackey space $(X, \tau)$, i.e. $\tau = \mu$, without property $P^0$ but containing a dense infinite codimensional subspace with property $P$, the three space problem has a negative solution for property $P$.

**Corollary 6.** The three space problem has a negative solution for $P$ semibornological [Mazur].

**Proof.** A slight modification of Example 3 of [3] provides an example of a barrelled (not Mazur) space $X$ containing a dense ultrabornological (hence semibornological and Mazur) subspace $G$ with $\dim(X/G) = \aleph_0$.

**References**

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