ASYMPTOTICALLY PERIODIC SOLUTIONS
OF A CLASS OF SECOND ORDER NONLINEAR
DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we give necessary and sufficient conditions for all solutions of the system

\[(S) \quad x' = y, \quad y' = -a(t)f(x)g(y)\]

to be oscillatory or bounded, for all orbits of the system

\[(S_1) \quad x' = y, \quad y' = -af(x)g(y)\]

to be periodic, where \(a(t) \to \alpha > 0\) as \(t \to \infty\), and for every orbit of (S) to approach a periodic orbit of (S_1). The conditions assuring that every solution of (S) is asymptotically periodic are also established.

1. Introduction. We consider the second order differential equation

\[(E) \quad x'' + a(t)f(x)g(x') = 0,\]

where \(a: I \to R^+ = (0, \infty), I = [\tau, \infty), f: R \to R = (-\infty, \infty)\) and \(g: R \to R^+\) are continuous functions, \(a(t) \to \alpha > 0\) as \(t \to \infty\), and \(xf(x) > 0\) for \(x \neq 0\). Assume that the solution of any Cauchy problem is unique and can be defined on \(I\).

We also consider the limit equation of (E),

\[(E_1) \quad x'' + af(x)g(x') = 0,\]

and the equivalent systems of (E) and (E_1),

\[(S) \quad x' = y, \quad y' = -a(t)f(x)g(y)\]

and

\[(S_1) \quad x' = y, \quad y' = -af(x)g(y).\]

The purpose of the present paper is to establish necessary and sufficient conditions for all solutions of (S) to be oscillatory or bounded, for all orbits of (S_1) to be periodic, and for every orbit of (S) to approach a periodic orbit of (S_1) as \(t \to \infty\), and also conditions to insure that every solution of (S) is asymptotically periodic. Our results improve and extend some theorems in [1–8].

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2. Equation (E).

THEOREM 1. (E) is oscillatory if and only if

\( H_1: \quad F(x) = \int_0^x f(u) \, du \to \infty \quad \text{as} \quad |x| \to \infty, \)

\( H_2: \quad G(y) = \int_0^y \frac{v}{g(v)} \, dv \to \infty \quad \text{as} \quad |y| \to \infty. \)

PROOF. Sufficiency follows directly from [3, Theorem 1].

Necessity. If \( F(\infty) < \infty \), then for given \( x'_0 > 0 \), there exists \( x_0 > 0 \) such that

\[
(1) \quad F(\infty) - F(x_0) < \frac{1}{\gamma} G(x'_0),
\]

where \( \gamma = \sup_{t \geq \tau} a(t) \). By assumption on \( a(t) \) clearly \( 0 < \gamma < \infty \).

Let \( x(t) \) be a solution of Cauchy problem (E) with \( x(\tau) = x_0, \ x'(\tau) = x'_0 \). By assumption, \( x(t) \) and its derivative \( x'(t) \) are oscillatory. Let \( T_1 \) be the first zero of \( x'(t) \) on \( I \). Clearly, \( \tau < T_1 < \infty \) and \( x(t) > 0 \) for \( t \in [\tau, T_1] \).

Integrating (E) from \( \tau \) to \( T_1 \), we get

\[
G(x'(T_1)) - G(x'(\tau)) = - \int_\tau^{T_1} a(t)f(x(t))x'(t) \, dt \geq -\gamma[F(x(T_1)) - F(x(\tau))].
\]

Then

\[
G(x'_0) \leq \gamma(F(\infty) - F(x_0)),
\]

which contradicts (1). For the case \( F(-\infty) < \infty \), the proof is similar, so \( H_1 \) is necessary.

If \( G(\infty) < \infty \), then for given \( x_0 < 0 \), there exists \( x'_0 > 0 \) such that

\[
(2) \quad G(\infty) - G(x'_0) < \beta F(x_0),
\]

where \( \beta = \inf_{t \geq \tau} a(t) \). Clearly \( 0 < \beta < \infty \).

Let \( x(t) \) be a solution of Cauchy problem (E) with \( x(\tau) = x_0, \ x'(\tau) = x'_0 \). By assumption, \( x(t) \) and \( x'(t) \) are oscillatory. Let \( T_2 \) be the first zero of \( x(t) \) on \( I \).

Clearly \( \tau < T_2 < \infty \) and \( x'(t) > 0 \) for \( t \in [\tau, T_2] \).

Integrating (E) from \( \tau \) to \( T_2 \), we get

\[
G(x'(T_2)) - G(x'(\tau)) = - \int_\tau^{T_2} a(t)f(x(t))x'(t) \, dt \geq -\beta[F(x(T_2)) - F(x(\tau))].
\]

Then

\[
G(\infty) - G(x'_0) \geq \beta F(x_0),
\]

which contradicts (2). For the case \( G(-\infty) < \infty \), the proof is similar, so \( H_2 \) is necessary. This completes the proof.

REMARK 1. The sufficient condition of Theorem 1 extends [1, Theorem 0.1 and 5, Theorem 2].

Let \( x(t) \) be a nontrivial oscillatory solution of (E). Clearly, \( x(t) \) and \( x'(t) = y(t) \) are oscillatory and their zeros separate one another. Let \( \{t_{2n}\} \) and \( \{t_{2n+1}\} \) be
sequences of zeros of \( x(t) \) and \( y(t) \) respectively such that \( t_{2n} < t_{2n+1} < t_{2n+2} \) (\( n = 0, 1, \ldots \)), \( t_n \to \infty \) as \( n \to \infty \).

Integrating (S) from \( t_{2k-1} \) to \( t_{2k} \), we obtain

\[ G(y(t_{2k})) = a(\tau_{2k-1})F(x(t_{2k-1})) \quad (t_{2k-1} < \tau_{2k-1} < t_{2k}, \ k = 1, 2, \ldots). \]

Similarly, integrating (S) from \( t_{2k} \) to \( t_{2k+1} \), we obtain

\[ G(y(t_{2k})) = a(\tau_{2k})F(x(t_{2k+1})) \quad (t_{2k} < \tau_{2k} < t_{2k+1}, \ k = 0, 1, \ldots). \]

Denote

\[ s_n = \prod_{k=1}^{n} \frac{a(\tau_{2k-1})}{a(\tau_{2k})}, \quad s'_n = \prod_{k=1}^{n} \frac{a(\tau_{2k-1})}{a(\tau_{2k-2})} \quad (n = 1, 2, \ldots). \]

The sequences \( \{s_n\} \) and \( \{s'_n\} \) are called characteristic sequences of the oscillatory solution \( x(t) \). Clearly,

\[ s'_n = \frac{a(\tau_{2n})}{a(\tau_0)} s_n \quad (n = 1, 2, \ldots). \]

We have the following lemma.

**Lemma 1.** Suppose that \( x(t) \) is a nontrivial oscillatory solution of (E). Then the sequences \( \{x(t_{2n+1})\} \) and \( \{x'(t_{2n})\} \) satisfy (3) and (4), the sequences \( \{s_n\} \) and \( \{s'_n\} \) satisfy (5).

**Lemma 2.** Suppose that all solutions of (E) are bounded. Then (E) is oscillatory.

Lemma 2 is an immediate corollary of [4, Theorem 1].

We need the following hypothesis:

H3: (E) is oscillatory and the characteristic sequence \( \{s_n\} \) of every nontrivial solution of (E) is bounded.

**Theorem 2.** All solutions of (E) and their derivatives are bounded if and only if \( H_1 - H_3 \) hold.

**Proof.** Sufficiency. By \( H_1, H_2, \) and Theorem 1, we know that any solution \( x(t) \) of (E) and \( x'(t) \) are oscillatory and hence by Lemma 1, (3)-(5) hold. From (3) and (4) we obtain

\[ F(x(t_{2n+1})) = s_n F(x(t_1)) \quad (n = 1, 2, \ldots) \]

and

\[ G(x'(t_{2n})) = s'_n G(x'(t_0)) \quad (n = 1, 2, \ldots). \]

From (6) and boundedness of the sequence \( \{s_n\} \), we get

\[ F(x(t_{2n+1})) \leq s F(x(t_1)) \quad (n = 1, 2, \ldots), \]

where \( s_n \leq s < \infty \). From this estimate and \( H_1 \) it follows that the sequence \( \{x(t_{2n+1})\} \) is bounded, so that \( x(t) \) is bounded.

From (5) and \( H_3 \), we have

\[ s'_n \leq \gamma s/a(\tau_0) \quad (n = 1, 2, \ldots), \]
where \( \gamma = \sup_{t \geq \tau} a(t) > 0 \), and in view of (7)

\[
G(x'(t_{2n})) \leq \frac{\gamma g}{\alpha(\tau_0)} G(x'(t_0)) \quad (n = 1, 2, \ldots).
\]

From this estimate and \( H_2 \) it follows that the sequence \( \{x'(t_{2n})\} \) is bounded, so that \( x'(t) \) is bounded.

**Necessity.** By Lemma 2 (E) is oscillatory, and hence by Theorem 1 \( H_1 \) and \( H_2 \) hold. Let \( x(t) \) be any bounded oscillatory solution of (E). From (6) we get

\[
s_n = F(x(t_{2n+1}))/F(x(t_1)) \leq L/F(x(t_1)) \quad (n = 1, 2, \ldots),
\]

where \( L = \max_{|x| \leq M} F(x) \) and \( M = \sup_{t \geq \tau} |x(t)| \). Thus \( \{s_n\} \) is bounded. This completes the proof.

**COROLLARY 1.** Suppose that the following holds.

\( H_4 : a(t) \) is a function of bounded variation on \( I \).

Then all solutions of (E) and their derivatives are bounded if and only if \( H_1 \) and \( H_2 \) hold.

In fact, under the assumption on \( a(t) \), it can be shown that every characteristic sequence \( \{s_n\} \) is bounded, for which it is sufficient to show the convergence of the infinite product

\[
s_\infty = \prod_{k=1}^{\infty} \frac{a(\tau_{2k-1})}{a(\tau_{2k})},
\]

which is equivalent to the convergence of the series

\[
\sigma = \sum_{k=1}^{\infty} \frac{a(\tau_{2k-1}) - a(\tau_{2k})}{a(\tau_{2k})}.
\]

By \( a(t) \to \alpha \) as \( t \to \infty \) and \( H_4 \), we have that

\[
\sum_{k=1}^{\infty} \frac{1}{\beta} \frac{|a(\tau_{2k-1}) - a(\tau_{2k})|}{a(\tau_{2k})} < \infty,
\]

where \( \beta = \inf_{t \geq \tau} a(t) > 0 \). This implies that the series \( \sigma \) is absolutely convergent and hence \( s_\infty \) is convergent. So \( \{s_n\} \) is bounded.

**REMARK 2.** The sufficiency condition of Corollary 1 improves and extends [2, Theorems 8 and 9; 7, Theorems 4 and 5; and 8, Theorems 1–3 and their corollary].

3. **Equation (E_1).** We note that there is one-to-one correspondence between periodic solutions of (E_1) and periodic orbits of (S_1), and that every orbit of (S_1) is round the origin.

**THEOREM 3.** Every solution of (E_1) is periodic if and only if \( H_1 \) and \( H_2 \) hold.

**Proof.** **Necessity.** Since every periodic solution of (E_1) is oscillatory, by Theorem 1 we have \( H_1 \) and \( H_2 \).

**Sufficiency.** Consider the function \( V(x, y) = \alpha F(x) + G(y) \). Taking the derivative of \( V(x, y) \) along an orbit of (S_1) we get

\[
V'(x, y) = \alpha f(x)y + (y/g(y))(-\alpha f(x)g(y)) = 0.
\]

Hence, from \( H_1 \) and \( H_2 \) it follows that \( V(x, y) = C \) is a closed orbit of (S_1) for each \( C > 0 \). This completes the proof.

**REMARK 3.** If \( g(y) = 1 \), then [1, Theorem 0.1 and 5, Theorem 2] can be derived from the sufficiency of Theorem 3.
4. Equations (E) and (E₁). In this and the next section we need the following hypothesis:

H₅: (E) is oscillatory and the characteristic sequence {sₙ} of every nontrivial solution of (E) satisfies sₙ → s > 0 as n → ∞.

**Theorem 4.** Every orbit of (S) approaches a periodic orbit of (S₁) in a spiral manner as t → ∞ if and only if H₁, H₂, and H₅ hold.

**Proof.** Sufficiency. By H₁, H₂, and Theorem 3 every orbit of (S₁) is closed. By Theorems 1 and 2 we know that every solution x(t) of (E) and x'(t) = y(t) are oscillatory and bounded, and hence (6), (7), and (5) hold. Letting n → ∞ in (6) and (7), by H₅ and (5) we have

\[ \lim_{n \to \infty} F(x(t_{2n+1})) = sF(x(t₁)) \]

and

\[ \lim_{n \to \infty} G(y(t_{2n})) = s'G(y(t₀)), \]

where s' = s₀/a(τ₀) > 0. Letting k → ∞ in (4), by (8) and (9) we find

\[ \alpha sF(x(t₁)) = s'G(y(t₀)) = C₁ > 0. \]

Consider the periodic orbit of (S₁):

\[ V(x, y) = \alpha F(x) + G(y) = C₁. \]

Clearly, V(x, y) = C₁ intersects the x-axis at (x₁, 0) and (x₂, 0), and the y-axis at (0, y₁) and (0, y₂). Without loss of generality we assume that x₁ > 0, x₂ < 0, y₁ > 0, and y₂ < 0, and x(t₄n₋₃) > 0, x(t₄n₋₁) < 0, y(t₄n₋₂) > 0, and y(t₄n) < 0. From (8)-(10) it follows that x(t₄n₋₃) → x₁, x(t₄n₋₁) → x₂, y(t₄n₋₂) → y₁, and y(t₄n) → y₂ as n → ∞. We now show that for arbitrary ε > 0, there exists τ' ≥ τ such that

\[ M(t) = (x(t), y(t)) \in \mathcal{A}(ε) = \{(x, y): C₁ - ε < V(x, y) < C₁ + ε\} \]

for all t ≥ τ'. In fact, assume the contrary. Then by boundedness of M(t), there exists a sequence \{t'j\}, t'j → ∞ as j → ∞, such that M(t'j) \notin \mathcal{A}(ε) and M(t'j) → M = (\bar{x}, \bar{y}) as j → ∞. Clearly, M \notin \mathcal{A}(ε). Without loss of generality we assume \bar{x} > 0, \bar{y} > 0. Then t₄nj₋₃ < t'j < t₄nj₋₂ for j sufficiently large. Integrating (S) from t₄nj₋₃ to t'j we obtain

\[ G(y(t'j)) + a(τ'₄nj₋₃)F(x(t'j)) = a(τ'₄nj₋₃)F(x(t₄nj₋₃)) \]

\[ (t₄nj₋₃ < t'₄nj₋₃ < t'j). \]

Letting j → ∞ in (11), we have V(\bar{x}, \bar{y}) = C₁, which is impossible.

Necessity. By assumption we know that any solution (x(t), y(t)) of (S) is oscillatory and bounded, and hence by Theorem 2 H₁ and H₂ hold.

As before, we assume x(t₄n₋₁) → x₁ > 0 and x(t₄n₋₁) → x₂ < 0 as n → ∞, and that V(x₁, 0) = V(x₂, 0) = C₁ > 0, from which we have F(x₁) = F(x₂) = C₁/α > 0. On the other hand, from (6) it follows that

\[ s₂k₋₂ = \frac{F(x(t₄k₋₃))}{F(x(t₁))}, \]
and
\[(13)\quad s_{2k-1} = F(x(t_{4k-1}))/F(x(t_1)).\]

Letting \(k \to \infty\) in (12) and (13), we obtain
\[
\lim_{k \to \infty} s_{2k-2} = F(x_1)/F(x(t_1)) = C_1/|\alpha F(x(t_1))|
\]
and
\[
\lim_{k \to \infty} s_{2k-1} = F(x_2)/F(x(t_1)) = C_1/|\alpha F(x(t_1))|.
\]

Then we have
\[
\lim_{n \to \infty} s_n = C_1/|\alpha F(x(t_1))| > 0.
\]

This completes the proof.

**Corollary 2.** Suppose that \(H_4\) holds. Then every orbit of (S) approaches a periodic orbit of (S\(_1\)) in a spiral manner as \(t \to \infty\) if and only if \(H_1\) and \(H_2\) hold.

In fact, as in the proof of Corollary 1, the infinite product \(s_\infty\) is convergent. From this and \(a(t) > 0\) for \(t \in I\) it follows that \(s_\infty > 0\). That is equivalent to \(s_n \to s_\infty\) as \(n \to \infty\) so that Corollary 2 can be derived from Theorem 4.

**Remark 4.** The sufficiency condition of Corollary 2 improves \([6, \text{Theorems 5 and 6}]\).

**5. Asymptotically periodic solutions.** A solution \((x(t), y(t))\) of (S) is defined to be asymptotically periodic if there exists a constant \(T > 0\) such that for arbitrary \(\epsilon > 0\), there is a \(\tau' \geq \tau\) so that
\[
|(x(t + T), y(t + T)) - (x(t), y(t))| < \epsilon\quad \text{for all } t \geq \tau
\]
(see \([6, \text{§5}]\)).

**Theorem 5.** Suppose that \(H_1, H_2,\) and \(H_5\) hold. Then every solution \((x(t), y(t))\) of (S) is asymptotically periodic.

**Proof.** As in the proof of Theorem 4, we assume that \(x(t_{4n-3}) \to x_1 > 0, x(t_{4n-1}) \to x_2 < 0, y(t_{4n-2}) \to y_1 > 0,\) and \(y(t_{4n}) \to y_2 < 0\) as \(n \to \infty\). Clearly, \((x(t + t_{4n-3}), y(t + t_{4n-3}))\) is a solution of the Cauchy problem
\[
u' = v, \quad v' = -\alpha f(u)g(v) + (\alpha - \alpha(a(t + t_{4n-3}))f(x(t + t_{4n-3}))g(y(t + t_{4n-3}))
\]
with \(u(0) = x(t_{4n-3}), v(0) = 0\). Let \((x^*(t), y^*(t))\) be a solution of the Cauchy problem (S\(_1\)) with \(x^*(0) = x_1, y^*(0) = 0\). By Theorem 3, \((x^*(t), y^*(t))\) is a periodic solution of (S\(_1\)) (periodic \(T > 0\)). Using Yoshizawa's method (see \([9, \text{§13}]\)), it is easy to prove that \((x(t + t_{4n-3}), y(t + t_{4n-3}))\) converges to \((x^*(t), y^*(t))\) uniformly in \(t \in [0, 3T]\) as \(n \to \infty\). Hence, for \(\epsilon > 0\), there exists an integer \(N > 0\) such that \(t_{4(n+1)-3} - t_{4n-3} < 2T\) and \(|(x(t + t_{4n-3}), y(t + t_{4n-3})) - (x^*(t), y^*(t))| < \epsilon/2\) for \(t \in [0, 3T]\) and \(n \geq N\). From this we have
\[
|(x(t + T + t_{4n-3}), y(t + T + t_{4n-3})) - (x(t + t_{4n-3}), y(t + t_{4n-3}))|
\leq |(x(t + T + t_{4n-3}), y(t + T + t_{4n-3})) - (x^*(t + T), y^*(t + T))|
+ |(x^*(t), y^*(t)) - (x(t + t_{4n-3}), y(t + t_{4n-3}))|
< \epsilon\quad \text{for } t \in [0, 2T].
\]
Furthermore, we obtain
\[ |(x(t + T), y(t + T)) - (x(t), y(t))| < \varepsilon \quad \text{for all} \quad t \geq t_{4N-3}. \]
This completes the proof.

**Corollary 3.** Suppose that \( H_1, H_2, \) and \( H_4 \) hold. Then the conclusion of Theorem 5 holds.

**Remark 5.** Corollary 3 improves [6, Theorem 7 and its Corollary].

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