PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 4, April 1987

AN EQUATION ALTERNATELY OF RETARDED AND ADVANCED TYPE

KENNETH L. COOKE AND JOSEPH WIENER

ABSTRACT. We study a differential equation with the argument $2[(t+1)/2]$, where $[\cdot]$ denotes the greatest-integer function. The argument deviation $\tau(t) = t - 2[(t+1)/2]$ is a function of period 2 and equals t for $-1 \leqslant t < 1$. It changes its sign in each interval $2n - 1 \leqslant t < 2n + 1$.

1. Introduction. This note continues the investigation of differential equations with piecewise constant arguments (EPCA) originated by K. L. Cooke and J. Wiener [**3**], and S. M. Shah and J. Wiener [**4**]. They are closely related to impulse and loaded equations and, especially, to difference equations of a discrete argument. These equations have the structure of continuous dynamical systems within intervals of certain length. Continuity of a solution at a point joining any two consecutive intervals then implies recursion relations for the solution at such points. The equations are thus similar in structure to those found in certain "sequential-continuous" models of disease dynamics as treated by S. Busenberg and K. L. Cooke [**2**]. The above works show that all types of EPCA share similar characteristics. First of all, it is natural to pose the initial-value problem for such equations not on an interval but a number of individual points. Secondly, two-sided solutions exist for all types of EPCA. Finally, since EPCA combine the features of both differential and difference equations, their asymptotic behavior as $t \to \infty$ resembles in some cases the growth of solutions of differential equations, while in others it inherits the properties of difference equations.

2. Main results. We consider the equation

$$x'(t) = ax(t) + a_0 x(2[(t+1)/2]), \qquad x(0) = c_0, \tag{1}$$

where $[\cdot]$ is the greatest-integer function. The argument deviation

$$\tau(t) = t - 2[(t+1)/2] \tag{2}$$

is negative for $2n - 1 \leqslant t < 2n$, and positive for $2n < t < 2n + 1$ (n is an integer).

Received by the editors February 10, 1986. Some of the contents of this article were presented at the International Congress of Mathematicians, Berkeley, California, August, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34K05, 34K15; Secondary 34K20.

The first author was partially supported by NSF Grant DMS-8405383.

The second author was partially supported by U. S. Army Research Grant DAAG29-84-G-0034.

Therefore, "equation (1)" is of considerable interest: on each interval $[2n-1, 2n+1)$ it is of alternately advanced and retarded type. "Equation (1)" is of advanced type on $[2n-1, 2n)$ and of retarded type on $(2n, 2n+1)$.

DEFINITION. A solution of "equation (1)" on $[0, \infty)$ is a function $x(t)$ that satisfies the conditions:

(i) $x(t)$ is continuous on $[0, \infty)$.

(ii) The derivative $x'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $t = 2n-1$ $(n = 1, 2, \dots)$, where one-sided derivatives exist.

(iii) "Equation (1)" is satisfied on each interval $2n-1 \leqslant t < 2n+1$.

In this paper, we show that problem (1) has a unique solution on $[0, \infty)$ and a unique backward solution on $(-\infty, 0]$. Also, we determine the set of (a, a_0) for which the zero solution is asymptotically stable as $t \to +\infty$, and the set of (a, a_0) such that all nontrivial solutions have no zeros in $(-\infty, \infty)$. The set of bounded solutions is characterized. Furthermore, the same equation with variable coefficients $a(t)$, $a_0(t)$ is examined, the condition for existence of a unique solution on $[0, \infty)$ is determined, and conditions are found under which all solutions are oscillatory.

Let

$$\lambda(t) = e^{at} + a^{-1}a_0(e^{at} - 1), \quad \lambda_1 = \lambda(1), \quad \lambda_{-1} = \lambda(-1). \tag{3}$$

THEOREM 1. *Problem* (1) *has on* $[0, \infty)$ *a unique solution*

$$x(t) = \lambda(\tau(t))(\lambda_1/\lambda_{-1})^{[(t+1)/2]}c_0, \quad \textit{if } \lambda_{-1} \neq 0, \tag{4}$$

where $\tau(t)$ *is given by* (2).

PROOF. Assuming that $x_n(t)$ is a solution of "equation (1)" on the interval $2n-1 \leqslant t < 2n+1$, with the condition $x_n(2n) = c_{2n}$, we have

$$x_n'(t) = ax_n(t) + a_0c_{2n}.$$

The general solution of this equation on the given interval is

$$x_n(t) = e^{a(t-2n)}c - a^{-1}a_0c_{2n},$$

with an arbitrary constant c. Putting here $t = 2n$ gives $c_{2n} = c - a^{-1}a_0c_{2n}$ and

$$x_n(t) = \lambda(t - 2n)c_{2n}. \tag{5}$$

For $t = 2n-1$, we have

$$x_n(2n-1) = c_{2n-1} = \lambda_{-1}c_{2n}, \qquad c_{2n} = \lambda_{-1}^{-1}c_{2n-1},$$

and for $t = 2n+1$,

$$x_n(2n+1) = c_{2n+1} = \lambda_1c_{2n}.$$

Hence,

$$c_{2n+1} = (\lambda_1/\lambda_{-1})c_{2n-1} = (\lambda_1/\lambda_{-1})^nc_1,$$
$$c_{2n} = \lambda_{-1}^{-1}c_{2n-1} = \lambda_{-1}^{-1}(\lambda_1/\lambda_{-1})^{n-1}c_1.$$

From (5) it follows that $x_0(t) = \lambda(t)c_0$ and $x_0(1) = c_1 = \lambda_1 c_0$ for $0 \leqslant t < 1$. Therefore, $c_{2n} = (\lambda_1/\lambda_{-1})^n c_0$ and

$$x_n(t) = \lambda(t - 2n)(\lambda_1/\lambda_{-1})^n c_0, \tag{6}$$

where $\lambda(t)$, λ_1, and λ_{-1} are given by (3). Formula (6) is equivalent to (4). It was obtained with the implicit assumption $a \neq 0$, but the limiting case of (4) as $a \to 0$ is the solution of problem (1) with $a = 0$, if $a_0 \neq 1$.

THEOREM 2. *The solution of problem* (1) *has a unique backward continuation on* $(-\infty, 0]$ *given by formula* (4) *if* $\lambda_1 \neq 0$.

PROOF. If $x_{-n}(t)$ denotes the solution of (1) on $-2n - 1 \leqslant t < -2n + 1$ satisfying the condition $x_{-n}(-2n) = c_{-2n}$, then from the equation $x'_{-n}(t) = ax_{-n}(t) + a_0 c_{-2n}$ it follows that

$$x_{-n}(t) = e^{a(t+2n)}c - a^{-1}a_0 c_{-2n}.$$

At $t = -2n$ we get

$$c = \left(1 + a^{-1}a_0\right)c_{-2n}, \qquad x_{-n}(t) = \lambda(t + 2n)c_{-2n}.$$

For $t = -2n + 1$,

$$x_{-n}(-2n + 1) = c_{-2n+1} = \lambda_1 c_{-2n}, \qquad c_{-2n} = \lambda_{-1}^{-1} c_{-2n+1}.$$

For $t = -2n - 1$,

$$x_{-n}(-2n - 1) = c_{-2n-1} = \lambda_{-1} c_{-2n}.$$

Hence,

$$c_{-2n-1} = (\lambda_{-1}/\lambda_1)c_{-2n+1} = (\lambda_{-1}/\lambda_1)^n c_{-1}$$

and

$$c_{-2n} = \lambda_{-1}^{-1}(\lambda_{-1}/\lambda_1)^{n-1} c_{-1}.$$

Finally, on $-1 \leqslant t < 0$ we have

$$x_0(t) = \lambda(t)c_0, \qquad x_0(-1) = c_{-1} = \lambda_{-1}c_0.$$

Thus, $c_{-2n} = (\lambda_{-1}/\lambda_1)^n c_0$ and

$$x_{-n}(t) = \lambda(t + 2n)(\lambda_{-1}/\lambda_1)^n c_0.$$

For $t \leqslant 0$, this formula coincides with (4).

THEOREM 3. *The solution* $x = 0$ *of "equation* (1)" *is asymptotically stable as* $t \to +\infty$ *if and only if* $|\lambda_1/\lambda_{-1}| < 1$.

PROOF. Since $|\tau(t)| \leqslant 1$ and $\lambda(\tau)$ is continuous, the function $\lambda(\tau(t))$ is bounded for all t. The proof then follows easily from (4).

THEOREM 4. *The solution* $x = 0$ *of "equation* (1)" *is asymptotically stable as* $t \to +\infty$ *if and only if any one of the following hypotheses is satisfied*:

(i) $$a < 0, \quad a_0 > -\frac{a(e^{2a}+1)}{(e^a-1)^2} \quad \text{or} \quad a_0 < -a;$$

(ii) $$a > 0, \quad -\frac{a(e^{2a}+1)}{(e^a-1)^2} < a_0 < -a;$$

(iii) $$a = 0, \quad a_0 < 0.$$

PROOF. For the function $\lambda(t)$ we have $\lambda'(t) = (a + a_0)e^{at}$. If $a + a_0 > 0$, then $\lambda(t)$ is increasing, and assuming $\lambda(-1) > 0$ leads to $\lambda(1) > \lambda(-1)$, that is, $\lambda_1/\lambda_{-1} > 1$. The conditions $a + a_0 > 0$ and $\lambda(-1) > 0$ can be written as $-a < a_0 < a/(e^a - 1)$. In this case, the solution $x = 0$ is unstable. The case $a + a_0 < 0$, $\lambda(-1) < 0$ is impossible. Indeed, the inequalities $a_0 < -a$ and $a_0 > a/(e^a - 1)$ are inconsistent because $-a < a/(e^a - 1)$. From $a + a_0 > 0$ and $\lambda(-1) < 0$ it follows that

(7) $$a_0 > a/(e^a - 1).$$

The inequality $\lambda_1/\lambda_{-1} < 1$ implies
$$e^a + \frac{a_0}{a}(e^a - 1) > e^{-a} + \frac{a_0}{a}(e^{-a} - 1),$$
which is equivalent to $a + a_0 > 0$. On the other hand, $\lambda_1/\lambda_{-1} > -1$ gives
$$e^a + \frac{a_0}{a}(e^a - 1) < -e^{-a} - \frac{a_0}{a}(e^{-a} - 1),$$
whence $(e^{2a} + 1)/(e^a - 1)^2 < -a_0/a$. If $a > 0$, then
$$a_0 < -\frac{a(e^{2a}+1)}{(e^a-1)^2}.$$
This contradicts (7). For $a < 0$, we have
$$a_0 > -\frac{a(e^{2a}+1)}{(e^a-1)^2}$$
and since $a/(e^a - 1) < -a(e^{2a} + 1)/(e^a - 1)^2$, hypothesis (i) ensures asymptotic stability of $x = 0$. Finally, the conditions $a + a_0 < 0$ and $\lambda(-1) > 0$ simply reduce to $a_0 < -a$. The same result follows from the inequality $\lambda_1 < \lambda_{-1}$. Furthermore, from $\lambda_1 > -\lambda_{-1}$ we obtain
$$-\frac{e^{2a}+1}{(e^a-1)^2} < \frac{a_0}{a}.$$
For $a > 0$, this confirms hypothesis (ii). The case $a < 0$ again leads to $a_0 < -a$. If $a = 0$, then $|\lambda_1/\lambda_{-1}| = |(1 + a_0)/(1 - a_0)| < 1$ holds for $a_0 < 0$.

THEOREM 5. *All nontrivial solutions of "equation* (1)" *have no zeros in* $(-\infty, \infty)$ *if and only if*

(8) $$-\frac{ae^a}{e^a-1} < a_0 < \frac{a}{e^a-1}.$$

PROOF. The conclusion follows from the fact that $\lambda(t)$ is a monotone function if $a + a_0 \neq 0$; then solution (4) has no zeros if and only if $\lambda(-1)$ and $\lambda(1)$ are of the same sign, that is, $\lambda_1/\lambda_{-1} > 0$. The case $\lambda_{-1} < 0$, $\lambda_1 < 0$ is impossible since it leads to inconsistent inequalities

$$a_0 < -\frac{ae^a}{e^a - 1}, \qquad a_0 > \frac{a}{e^a - 1}.$$

On the contrary, the case $\lambda_{-1} > 0$, $\lambda_1 > 0$ yields (8). If $a + a_0 = 0$, the only solution of problem (1) is $x(t) = c_0$.

THEOREM 6. *The problem*

$$x'(t) = a(t)x(t) + a_0(t)x(2[(t+1)/2]), \qquad x(0) = c_0 \tag{9}$$

has a unique solution on $[0, \infty)$ *if* $a(t)$ *and* $a_0(t)$ *are continuous for* $t \geqslant 0$, *and*

$$\int_{2n-1}^{2n} u^{-1}(t)a_0(t)\,dt \neq u^{-1}(2n), \qquad n = 1, 2, \ldots,$$

where u^{-1} *is the reciprocal of* u *and* $u(t) = \exp(\int_0^t a(s)\,ds)$.

THEOREM 7. *The functional differential inequality*

$$x'(t) + p(t)x(t) + q(t)x(2[(t+1)/2]) \leqslant 0 \tag{10}$$

with $p(t)$ *and* $q(t)$ *continuous on* $[0, \infty)$ *has no eventually positive solution if*

$$\limsup_{n\to\infty} \int_{2n}^{2n+1} q(t)\exp\left(\int_{2n}^{t} p(s)\,ds\right) dt > 1. \tag{11}$$

PROOF. Following [**1**], we prove that the existence of an eventually positive solution leads to a contradiction. To this end suppose that $x(t)$ is a solution of (10) such that $x(t) > 0$ for $t \geqslant 2n$, where n is a sufficiently large integer. For $2n - 1 \leqslant t < 2n + 1$, inequality (10) becomes

$$x'(t) + p(t)x(t) + q(t)x(2n) \leqslant 0,$$

or

$$y'(t) + q(t)\exp\left(\int_{2n}^{t} p(s)\,ds\right) y(2n) \leqslant 0, \tag{12}$$

where $y(t) = x(t)\exp(\int_{2n}^t p(s)\,ds)$. Integrating (12) from $2n$ to $2n + 1$, we have

$$y(2n+1) \leqslant y(2n)\left(1 - \int_{2n}^{2n+1} q(t)\exp\left(\int_{2n}^{t} p(s)\,ds\right) dt\right).$$

Since $y(t) > 0$ for $t \geqslant 2n$, then

$$1 - \int_{2n}^{2n+1} q(t)\exp\left(\int_{2n}^{t} p(s)\,ds\right) dt > 0,$$

or

$$\limsup_{n\to\infty} \int_{2n}^{2n+1} q(t)\exp\left(\int_{2n}^{t} p(s)\,ds\right) dt \leqslant 1.$$

This contradicts (11). So, (10) has no eventually positive solution.

THEOREM 8. *If condition* (11) *is satisfied, the functional differential inequality*

$$x'(t) + p(t)x(t) + q(t)x(2[(t+1)/2]) \geqslant 0 \tag{13}$$

has no eventually negative solution.

From Theorems 7 and 8 it follows that subject to hypothesis (11), the equation

$$x'(t) + p(t)x(t) + q(t)x(2[(t+1)/2]) = 0 \tag{14}$$

has no eventually positive or eventually negative solutions and therefore we are led to the following conclusion.

THEOREM 9. *Subject to condition* (11), "*equation* (14)" *has oscillatory solutions only.*

COROLLARY. "*Equation* (9)" *has only oscillatory solutions on* $[0, \infty)$ *if*

$$\liminf_{n\to\infty} \int_{2n}^{2n+1} a_0(t) \exp\left(-\int_{2n}^{t} a(s)\,ds\right) dt < -1. \tag{15}$$

REMARK. Condition (15) is sharp. For "equation (1)" with constant coefficients, (15) becomes $a_0 < -ae^a/(e^a - 1)$ which is, according to (8), one of the two "best possible" conditions for oscillation.

THEOREM 10. *Inequality* (10) *has no eventually negative solution if*

$$\liminf_{n\to\infty} \int_{2n-1}^{2n} q(t) \exp\left(\int_{2n}^{t} p(s)\,ds\right) dt < -1. \tag{16}$$

PROOF. Suppose that $x(t)$ is a solution of (10) such that $x(t) < 0$ for $t \geqslant 2n - 1$, where n is a sufficiently large integer. Integrating (12) from $2n - 1$ to $2n$ gives

$$y(2n)\left(1 + \int_{2n-1}^{2n} q(t) \exp\left(\int_{2n}^{t} p(s)\,ds\right) dt\right) \leqslant y(2n-1),$$

and since $y(t) < 0$ for $t \geqslant 2n - 1$, then

$$1 + \int_{2n-1}^{2n} q(t) \exp\left(\int_{2n}^{t} p(s)\,ds\right) dt > 0,$$

or

$$\liminf_{n\to\infty} \int_{2n-1}^{2n} q(t) \exp\left(\int_{2n}^{t} p(s)\,ds\right) dt \geqslant -1,$$

which contradicts (16).

THEOREM 11. *If condition* (16) *is satisfied,* (13) *has no eventually positive solution.*

THEOREM 12. *Subject to condition* (16), "*equation* (14)" *has oscillatory solutions only.*

COROLLARY. "*Equation* (9)" *has only oscillatory solutions on* $[0, \infty)$ *if*

$$\limsup_{n\to\infty} \int_{2n-1}^{2n} a_0(t) \exp\left(-\int_{2n}^{t} a(s)\,ds\right) dt > 1. \tag{17}$$

REMARK. Condition (17) is sharp. For "equation (1)" with constant coefficients, (17) becomes $a_0 > a/(e^a - 1)$ which is, according to (8), one of the two "best possible" conditions for oscillation.

THEOREM 13. *If* $a_0 > a/(e^a - 1)$, *then solution* (4) *with the condition* $x(0) = c_0$ *has precisely one zero in each interval* $2n - 1 < t < 2n$ *with integral endpoints. If* $a_0 < -ae^a/(e^a - 1)$, *then* (4) *has precisely one zero in each interval* $2n < t < 2n + 1$.

THEOREM 14. *All solutions of "equation* (1)" *that are bounded on* $-\infty < t < \infty$ *and that do not tend to zero as* $t \to \pm\infty$ *are periodic. They exist only for* $a_0 = -a$ *or* $a_0 = -a(e^{2a} + 1)/(e^a - 1)^2$. *In the first case, the solutions are constant; and in the second case, they are of period* 4.

PROOF. The conclusions follow from (4) and from the condition $|\lambda_1/\lambda_{-1}| = 1$ which is necessary and sufficient for $x(t)$ to be bounded and not vanish.

REFERENCES

1. A. R. Aftabizadeh and J. Wiener, *Oscillatory properties of first order linear functional differential equations*, Applicable Anal. **20** (1985), 165–187.

2. S. Busenberg and K. L. Cooke, *Models of vertically transmitted diseases with sequential-continuous dynamics*, Nonlinear Phenomena in Mathematical Sciences (V. Lakshmikantham, Ed.), Academic Press, New York, 1982, pp. 179–187.

3. K. L. Cooke and J. Wiener, *Retarded differential equations with piecewise constant delays*, J. Math. Anal. Appl. **99** (1984), 265–297.

4. S. M. Shah and J. Wiener, *Advanced differential equations with piecewise constant argument deviations*, Internat. J. Math. Math. Sci. **6** (1983), 671–703.

DEPARTMENT OF MATHEMATICS, POMONA COLLEGE, CLAREMONT, CALIFORNIA 91711

DEPARTMENT OF MATHEMATICS, PAN AMERICAN UNIVERSITY, EDINBURG, TEXAS 78539