ON TOTALLY REAL 3-DIMENSIONAL SUBMANIFOLDS OF THE NEARLY KAHLER 6-SPHERE

F. DILLEN, B. OPOZDA, L. VERSTRAELEN AND L. VRANCKEN

Abstract. Let $M$ be a compact 3-dimensional totally real submanifold of the nearly Kaehler 6-dimensional unit sphere. Let $K$ be the sectional curvature function of $M$. Then, if $K > 1/16$, $M$ is a totally geodesic submanifold (and $K = 1$).

1. Introduction. On a 6-dimensional unit sphere $S^6(1)$, one can construct a nearly Kaehler structure $J$ making use of the Cayley number system. We recall this construction in §3.

In this paper we study 3-dimensional totally real submanifolds of $S^6(1)$. The definition and the basic formulas for such manifolds are given in §4. N. Ejiri [E] proved the following: If a 3-dimensional totally real submanifold of $S^6(1)$ has constant curvature $K$, then $K = 1$ or $K = \frac{1}{16}$. The main purpose of this article is the following theorem, which will be proved in §5.

Theorem. Let $M$ be a compact 3-dimensional totally real submanifold of $S^6(1)$. If all sectional curvatures $K$ of $M$ satisfy $\frac{1}{16} < K \leq 1$, then $K = 1$ on $M$.

The proof is based on integral formulas for Riemannian manifolds of A. Ros [R], which are given in §2.

2. Integral formulas. Let $M$ be a compact Riemannian manifold, $UM$ its unit tangent bundle, and $UM_p$ the fiber of $UM$ over a point $p$ of $M$. We denote by $dp$, $du$, and $du_p$ respectively the canonical measures on $M$, $UM$, and $UM_p$.

For any continuous function $f: UM \to \mathbb{R}$ one has

$$\int_{UM} f du = \int_M \left( \int_{UM_p} f du_p \right) dp.$$ 

Let $T$ be any $k$-covariant tensor field on $M$. Then the integral formulas state that

$$\int_{UM} (\nabla T)(u, u, u, \ldots, u) du = 0$$

and

$$\int_{UM} \sum_i (\nabla T)(e_i, e_i, u, u, \ldots, u) du = 0,$$

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where \( \{ e_i \}_{i=1}^n \) is an orthonormal basis of \( TM \), the tangent bundle over \( M \), and \( \nabla \) denotes the Levi Civita connection of \( M \).

### 3. The nearly Kaehler \( S^6(1) \)

Let \( e_0, e_1, \ldots, e_7 \) be the standard basis of \( \mathbb{R}^8 \). Then each point \( \alpha \) of \( \mathbb{R}^8 \) can be written in a unique way as \( \alpha = Ae_0 + x \), where \( A \in \mathbb{R} \) and \( x \) is a linear combination of \( e_1, \ldots, e_7 \). \( \alpha \) can be viewed as a Cayley number, and is called purely imaginary when \( A = 0 \). For any pair of purely imaginary \( x \) and \( y \), we consider the multiplication \( \cdot \) given by

\[
x \cdot y = -\langle x, y \rangle e_0 + x \times y,
\]

where \( \langle \quad, \quad \rangle \) is the standard scalar product on \( \mathbb{R}^8 \) and \( x \times y \) is defined by the following multiplication table for \( e_j \times e_k \):

<table>
<thead>
<tr>
<th>( j/k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>( e_3 )</td>
<td>(-e_2)</td>
<td>( e_5 )</td>
<td>(-e_4)</td>
<td>( e_7 )</td>
<td>(-e_6)</td>
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<tr>
<td>2</td>
<td>(-e_3)</td>
<td>0</td>
<td>( e_1)</td>
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<td>3</td>
<td>( e_2 )</td>
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<td>0</td>
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</tbody>
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For two Cayley numbers \( \alpha = Ae_0 + x \) and \( \beta = Be_0 + y \), the Cayley multiplication \( \cdot \), which makes \( \mathbb{R}^8 \) the Cayley algebra \( \mathcal{C} \), is defined by

\[
\alpha \cdot \beta = ABe_0 + Ay + Bx + x \cdot y.
\]

We recall that the multiplication \( \cdot \) of \( \mathcal{C} \) is neither commutative nor associative. The set \( \mathcal{C}_+ \) of all purely imaginary Cayley numbers clearly can be viewed as a 7-dimensional linear subspace \( \mathbb{R}^7 \) of \( \mathbb{R}^8 \). In \( \mathcal{C}_+ \) we consider the unit hypersphere which is centered at the origin:

\[
S^6(1) = \{ x \in \mathcal{C}_+ | \langle x, x \rangle = 1 \}.
\]

Then the tangent space \( T_xS^6 \) of \( S^6(1) \) at a point \( x \) may be identified with the affine subspace of \( \mathcal{C}_+ \) which is orthogonal to \( x \).

On \( S^6(1) \) we now define a \( (1, 1) \)-tensor field \( J \) by putting

\[
J_U = x \times U,
\]

where \( x \in S^6(1) \) and \( U \in T_xS^6 \). This tensor field is well defined (i.e., \( J_U \in T_xS^6 \)) and determines an almost complex structure on \( S^6(1) \), i.e. \( J^2 = -Id \), where \( Id \) is the identity transformation [F]. The compact simple Lie group \( G_2 \) is the group of automorphisms of \( \mathcal{C} \) and acts transitively on \( S^6(1) \) and preserves both \( J \) and the standard metric on \( S^6(1) \) [FI].

Further, let \( G \) be the \( (2, 1) \)-tensor field on \( S^6(1) \) defined by

\[
(3.1) \quad G(X, Y) = (\tilde{\nabla}_X J)Y,
\]

where \( X, Y \in \mathfrak{X}(S^6) \) and where \( \tilde{\nabla} \) is the Levi Civita connection on \( S^6(1) \). This tensor field has the following properties:

\[
(3.2) \quad G(X, X) = 0,
\]
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(3.3) \[ G(X, Y) + G(Y, X) = 0, \]
(3.4) \[ G(X, JY) + JG(X, Y) = 0, \]
(3.5) \[ \langle \tilde{\nabla}_X G(Y, Z) \rangle = \langle Y, JZ \rangle X + \langle X, Z \rangle JY - \langle X, Y \rangle JZ, \]
(3.6) \[ \langle G(X, Y), Z \rangle + \langle G(X, Z), Y \rangle = 0, \]
(3.7) \[ \langle G(X, Y), G(Z, W) \rangle = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle \]
\[ + \langle JX, Z \rangle \langle Y, JW \rangle - \langle JX, W \rangle \langle Y, JZ \rangle, \]
where \( X, Y, Z, W \in \mathcal{F}(S^6) \). We recall that (3.2) means that the structure \( J \) is nearly Kaehler, i.e. \( \forall X \in \mathcal{F}(S^6): (\tilde{\nabla}_X J) X = 0. \)

4. Totally real submanifolds of \( S^6 \). A Riemannian manifold \( M \), isometrically immersed in \( S^6 \), is called a totally real submanifold of \( S^6 \) if \( J(TM) \subseteq T^\perp M \), where \( T^\perp M \) is the normal bundle of \( M \) in \( S^6 \). Then, we have \( \dim M \leq 3 \). In this paper we consider the case \( \dim M = 3 \). In [E] Ejiri proved that a 3-dimensional totally real submanifold of \( S^6 \) is orientable and minimal, and that \( G(X, Y) \) is orthogonal to \( M \) for \( X, Y \in \mathcal{F}(M) \). We denote the Levi Civita connection of \( M \) by \( \nabla \). The formulas of Gauss and Weingarten are then given by

\[ \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \] (4.1)

and

\[ \tilde{\nabla}_X \xi = -A_{\xi} X + D_X \xi, \] (4.2)

where \( X \) and \( Y \) are vector fields on \( M \) and \( \xi \) is a normal vector field on \( M \). The second fundamental form \( h \) is related to \( A_{\xi} \) by

\[ \langle h(X, Y), \xi \rangle = \langle A_{\xi} X, Y \rangle. \] (4.3)

From (4.1) and (4.2) we find

\[ D_X (JY) = G(X, Y) + J\nabla_X Y \] (4.4)

and

\[ A_{JX} Y = -Jh(X, Y). \] (4.5)

If we denote the curvature tensors of \( \nabla \) and \( D \) by \( R \) and \( R^D \), respectively, then the equations of Gauss, Codazzi, and Ricci are given by

\[ R(X, Y, Z, W) = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle \]
\[ + \langle h(X, Z), h(Y, W) \rangle - \langle h(X, W), h(Y, Z) \rangle, \] (4.6)

(\[ \nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z), \] (4.7)

\[ \langle R^D(X, Y) \xi, \mu \rangle = \langle [A_{\xi}, A_{\mu}] X, Y \rangle, \] (4.8)

where \( X, Y, Z, W \in \mathcal{F}(M) \), \( \xi \) and \( \mu \) are normal vector fields, and \( \nabla h \) is defined by

\[ \nabla h(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \]

From (4.5), (4.6), and (4.8) we obtain

\[ \langle R^D(X, Y) JZ, JW \rangle = \langle R(X, Y) Z, W \rangle + \langle Z, X \rangle \langle Y, W \rangle - \langle Z, Y \rangle \langle X, W \rangle. \] (4.9)
We also define $\nabla^2 h$ by

$$
- \nabla h(Y, \nabla_X Z, W) - h(Y, Z, \nabla_X W).
$$

Then $\nabla^2 h$ satisfies the following equation:

$$
- h(R(X, Y)Z, W) - h(Z, R(X, Y)W).
$$

5. Proof of the theorem. In this section, $X, Y, Z, \ldots$ denote tangent vector fields on a 3-dimensional totally real submanifold $M$ of $S^6$, with second fundamental tensor $h$. Then by straightforward computations one may prove the following.

**Lemma 1.**

(a) $\langle h(X, Y), JZ \rangle = \langle h(X, Z), JY \rangle$,

(b) $\langle (\nabla h)(X, Y, Z), JW \rangle = \langle (\nabla h)(X, Y, W), JZ \rangle - \langle h(Y, Z), G(X, W) \rangle
- \langle h(Y, W), G(X, Z) \rangle$,

(c) $\langle (\nabla^2 h)(X, Y, Z, W), JU \rangle = \langle (\nabla^2 h)(X, Y, Z, U), JW \rangle
- \langle (\nabla h)(X, Z, W), G(Y, U) \rangle + \langle (\nabla h)(X, Z, U), G(Y, W) \rangle
- \langle h(Z, W), (\tilde{\nabla} G)(X, Y, U) \rangle + \langle h(Z, U), (\tilde{\nabla} G)(X, Y, W) \rangle
+ \langle (\nabla h)(Y, Z, W), G(X, U) \rangle - \langle \nabla h(Y, Z, W), G(X, U) \rangle$.

Now let $v \in UM_p$, $p \in M$. If $e_2$ and $e_3$ are orthonormal vectors in $UM_p$, orthogonal to $v$, then we can consider $\{e_2, e_3\}$ as an orthonormal basis of $T_v(UM_p)$. We remark that $\{v, e_2, e_3\}$ is an orthonormal basis of $T_p M$. We can choose $e_3$ such that $G(v, e_2) = J e_3$, $G(e_2, e_3) = J v$, and $G(e_3, v) = e_2$ (cf. [E]). If we denote the Laplacian of $UM_p \equiv S^2$ by $\Delta$, then $\Delta f = e_2 e_2 f + e_3 e_3 f$, where $f$ is a differentiable function on $UM_p$.

**Lemma 2.**

$$
3 \int_{UM_p} \| h(v, v) \|^2 = 7 \int_{UM_p} \langle h(v, v), Jv \rangle^2.
$$

**Proof.** Define a function $f$ on $UM_p$, $p \in M$, by $f(v) = \langle h(v, v), Jv \rangle^2$. Using Lemma 1(a) and the minimality of $M$ we can prove that

$$
(\Delta f)(v) = -42 f(v) + 18 \| h(v, v) \|^2.
$$

Integrating this completes the proof. \[\square\]

**Lemma 3.**

$$
\int_{UM} \sum_{i=1}^3 \langle (\nabla h)(e_i, v, v), Jv \rangle \langle G(v, e_i), h(v, v) \rangle = \frac{1}{3} \int_{UM} \langle h(v, v), Jv \rangle^2,
$$

where $\{e_1, e_2, e_3\}$ is an arbitrary orthonormal basis of $TM$. 

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PROOF. Define the covariant tensor field $T_1$ by

$$T_1(X_1, X_2, \ldots, X_7) = \langle G(X_1, X_2), h(X_3, X_4) \rangle \langle h(X_5, X_6), JX_7 \rangle.$$ 

Then

$$\int_{U_M} \sum_{i=1}^{3} \langle \nabla T_1 \rangle (e_i, v, v, v, v, v, v) = 0. \tag{5.1}$$

To compute $\sum_{i=1}^{3} \langle \nabla T_1 \rangle (e_i, v, v, v, v, v, v)$, we can choose an orthonormal basis $\{e_1, e_2, e_3\}$ such that $e_1 = v$, $G(e_1, e_2) = Je_3$, $G(e_2, e_3) = Je_1$, and $G(e_3, e_1) = Je_2$. This is allowed because $\sum_{i=1}^{3} \langle \nabla T_1 \rangle (e_i, e_i, v, v, v, v, v, v)$ does not depend on the choice of the basis. (Similar argumentations will be used through this paper, without mentioning it.) Since

$$\sum_{i=1}^{3} \langle G(e_i, v) \rangle \langle (\nabla h)(e_i, v, v) \rangle = \langle h(v, v), Jv \rangle, \tag{5.2}$$

we find that

$$\sum_{i=1}^{3} \langle \nabla T_1 \rangle (e_i, e_i, v, v, v, v, v) = \|h(v, v)\|^2 - 2 \langle h(v, v), Jv \rangle^2$$

$$+ \sum_{i=3}^{3} \langle (\nabla h)(e_i, v, v), Jv \rangle \langle G(e_i, v), h(v, v) \rangle.$$

Lemma 3 then follows from (5.1) and Lemma 2. \qed

Because of Lemma 1(b), we have

$$\| (\nabla h)(v, v, v) \|^2 = \sum_{i=1}^{3} \langle (\nabla h)(e_i, v, v), Jv \rangle^2$$

$$- 2 \sum_{i=1}^{3} \langle (\nabla h)(v, v, e_i), Jv \rangle \langle h(v, v), G(v, e_i) \rangle$$

$$+ \|h(v, v)\|^2 - \langle h(v, v), Jv \rangle^2.$$ 

Integrating this and using Lemmas 2 and 3, we also obtain the following.

**Lemma 4.**

$$\int_{U_M} \| (\nabla h)(v, v, v) \|^2$$

$$= \int_{U_M} \sum_{i=1}^{3} \langle (\nabla h)(e_i, v, v), Jv \rangle^2 + \frac{2}{3} \int_{U_M} \langle h(v, v), Jv \rangle^2. \tag{\text{\textdagger}}$$

**Lemma 5.**

$$\int_{U_M} \sum_{i=1}^{3} \langle (\nabla h)(e_i, v, v), Jv \rangle^2 = \frac{9}{4} \int_{U_M} \langle (\nabla h)(v, v, v), Jv \rangle^2$$

$$+ \frac{1}{12} \int_{U_M} \langle h(v, v), Jv \rangle^2.$$
Proof. Define the function $h$ on $UM_p$, $p \in M$, by $h(v) = \langle (\nabla h)(v, v, v), Jv \rangle^2$. Then we obtain

$$(\Delta h)(v) = -72h(v) + 2\|(\nabla h)(v, v, v)\|^2 + 30\sum_{i=1}^{3} \langle (\nabla h)(e_i, v, v), Jv \rangle^2$$

$$-12\sum_{i=1}^{3} \langle \nabla h(e_i, v, v), Jv \rangle \langle G(v, e_i), h(v, v) \rangle.$$

Integrating this over $UM$, using Lemmas 3 and 4, we obtain Lemma 5. □

Combining Lemmas 4 and 5 we obtain

Lemma 6.

$$\int_{UM} \|(\nabla h)(v, v, v)\|^2$$

$$= \frac{9}{4} \int_{UM} \langle (\nabla h)(v, v, v), Jv \rangle^2 + \frac{3}{4} \int_{UM} \langle h(v, v), Jv \rangle^2. \quad \square$$

Lemma 7.

$$\int_{UM} \langle (\nabla h)(v, v, v), Jv \rangle^2 + \int_{UM} \langle (\nabla^2 h)(v, v, v), Jv \rangle \langle h(v, v), Jv \rangle = 0.$$

Proof. Define $T_2$ by

$$(5.3) \quad T_2(X_1, X_2, \ldots, X_6) = \langle h(X_1, X_2), JX_3 \rangle \langle h(X_4, X_5), JX_6 \rangle.$$

We know that

$$\int_{UM} \langle \nabla^2 T_2 \rangle(v, v, v, v, v, v, v, v) = 0.$$

Since

$$\langle \nabla^2 T_2 \rangle(v, v, v, v, v, v, v, v) = 2\langle (\nabla^2 h)(v, v, v), Jv \rangle \langle h(v, v), Jv \rangle$$

$$+ 2\langle (\nabla h)(v, v, v), Jv \rangle^2,$$

Lemma 7 is proved. □

Lemma 8.

$$\int_{UM} \|(\nabla h)(v, v, v)\|^2 + \int_{UM} \sum_{i=1}^{3} \langle (\nabla^2 h)(e_i, e_i, v, v), Jv \rangle \langle h(v, v), Jv \rangle = 0.$$

Proof. Define $T_2$ by (5.3). We know that

$$\int_{UM} \sum_{i=1}^{3} \langle \nabla^2 T_2 \rangle(e_i, e_i, v, v, v, v, v, v) = 0.$$
By a straightforward computation, we can prove that
\[
\left(\nabla^2 T_2\right)(e_i, e_i, v, v, v, v, v, v)
= 2\left(\left(\nabla h\right)(v, v), J\right)^2 + 2\left(\left(\nabla^2 h\right)(e_i, e_i, v, v), J\right)\langle h(v, v), J\rangle
+ 4\left(\left(\nabla h\right)(e_i, v, v), G(e_i, v)\right)\langle h(v, v), J\rangle
+ 2\left(h(v, v), \left(\nabla G\right)(e_i, e_i, v)\right)\langle h(v, v), J\rangle.
\]
Because of (3.1) we have
\[
\sum_{i=1}^3 \langle h(v, v), \left(\nabla G\right)(e_i, e_i, v)\rangle \langle h(v, v), J\rangle = -2\langle h(v, v), J\rangle^2.
\]
Using (5.4) we thus obtain
\[
\sum_{i=1}^3 \left(\nabla^2 T_2\right)(e_i, e_i, v, v, v, v, v, v)
= 2\left(\left(\nabla^2 h\right)(e_i, e_i, v, v), J\right)\langle h(v, v), J\rangle
+ 2\|\left(\nabla h\right)(v, v)\|^2.
\]
Integrating this completes the proof. □

**Lemma 9.**
\[
\int_{UM} \|\left(\nabla h\right)(v, v, v)\|^2 + \int_{UM} \left(\left(\nabla^2 h\right)(v, v, v, v), h(v, v)\right) = 0.
\]

**Proof.** Define the covariant tensor field $T_3$ on $M$ by
\[
T_3(X_1, X_2, X_3, X_4) = \langle h(X_1, X_2), h(X_3, X_4)\rangle.
\]
We know that
\[
\int_{UM} \left(\nabla^2 T_3\right)(v, v, v, v, v, v) = 0.
\]
Because
\[
\left(\nabla^2 T_3\right)(v, v, v, v, v, v) = 2\left(\left(\nabla^2 h\right)(v, v, v, v), h(v, v)\right) + 2\|\nabla h(v, v)\|^2,
\]
Lemma 9 follows. □

**Lemma 10.**
\[
\frac{3}{4} \int_{UM} \langle \left(\nabla h\right)(v, v, v), J\rangle^2
- \frac{1}{12} \int_{UM} \langle h(v, v), J\rangle^2 + \int_{UM} R(v, A_{J_0^p}, A_{J_0^p}, v) = 0.
\]

**Proof.** Define the function $g$ on $UM_p$, $p \in M$, by
\[
g(v) = \langle h(v, v), J\rangle \left(\left(\nabla^2 h\right)(v, v), J\right)\langle h(v, v), J\rangle.
\]
By a computation using (4.3), (4.4), the minimality of $M$, (5.2), and several times Lemma 1, we can prove that
\[
(\Delta g)(v) = -72g(v) + 30\left\langle h(v, v), (\nabla^2 h)(v, v, v, v) \right\rangle \\
- 24\left\langle h(v, v), Jv \right\rangle^2 + 30\left\| h(v, v) \right\|^2 \\
- 48\sum_{i=1}^{3} \left\langle (\nabla h)(v, v, e_i), Jv \right\rangle \left\langle h(v, v), G(v, e_i) \right\rangle \\
- 18R(v, A_{Jp}v, A_{Jp}v, v) \\
+ 8\sum_{i=1}^{3} \left\langle h(v, v), Jv \right\rangle \left\langle (\nabla^2 h)(e_i, e_i, v, v), Jv \right\rangle.
\]
Integrating this over $UM$, using Lemmas 7, 9, 2, 3, and 8, we obtain
\[
72\int_{UM} \langle (\nabla h)(v, v, v), Jv \rangle + 30\int_{UM} \langle h(v, v), Jv \rangle^2 \\
- 38\int_{UM} \left\| (\nabla h)(v, v, v) \right\|^2 - 18\int_{UM} R(v, A_{Jp}v, A_{Jp}v, v) = 0.
\]
Lemma 6 then completes the proof. □

Making use of Lemmas 2 and 4 we can rewrite Lemma 10 as follows.

**Lemma 11.**
\[
\frac{3}{4} \int_{UM} \langle (\nabla h)(v, v, v), Jv \rangle^2 \\
+ \int_{UM} \left[ R(v, A_{Jp}v, A_{Jp}v, v) - \frac{1}{16} \left( \left\| A_{Jp}v \right\|^2 - \langle A_{Jp}v, v \rangle^2 \right) \right] = 0. □
\]

**Proposition.** If $M$ is a 3-dimensional compact totally real submanifold of $S^6$ and if all sectional curvatures $K$ of $M$ satisfy $K \geq \frac{1}{16}$, then

1. $\langle (\nabla h)(v, v, v), Jv \rangle = 0$, and
2. $R(v, A_{Jp}v, A_{Jp}v, v) = \frac{1}{16}(\left\| A_{Jp}v \right\|^2 - \langle A_{Jp}v, v \rangle^2)$, for all $p \in M$ and $v \in UM_p$.

**Proof.** Under the assumptions of the proposition,
\[
R(v, A_{Jp}v, A_{Jp}v, v) - \frac{1}{16} \left( \left\| A_{Jp}v \right\|^2 - \langle A_{Jp}v, v \rangle^2 \right) \geq 0,
\]
and Lemma 11 then implies the proposition. □

Now we can prove the theorem. Suppose that all sectional curvatures $K$ of $M$ satisfy $K > \frac{1}{16}$. By the proposition, we have
\[
R(v, A_{Jp}v, A_{Jp}v) = \frac{1}{16} \left( \left\| A_{Jp}v \right\|^2 - \langle A_{Jp}v, v \rangle^2 \right).
\]
If there exist a unit vector $v \in UM_p$, $p \in M$, such that $A_{Jp}v$ is not parallel to $v$, then $v$ and $A_{Jp}v$ determine a plane of which the sectional curvature equals $\frac{1}{16}$ according to (5.5). This is a contradiction. Therefore all $A_{Jp}v$ are parallel to $v$. This implies because of (4.5) that
\[
\left\| h(v, v) \right\|^2 = \langle h(v, v), Jv \rangle^2.
\]
for all $v \in UM$. (5.6) together with Lemma 2 implies that $h = 0$, i.e., that $M$ is totally geodesic. This completes the proof.

6. Examples. 1. Let $M = \{ x \in S^6 \mid x = x_1e_1 + x_2e_3 + x_4e_5 + x_7e_7 \}$, and let $i$ be the inclusion map from $M$ into $S^6$. Then $(M, i)$ is a 3-dimensional totally real and totally geodesic submanifold of $S^6$.

2. In [E] Ejiri announced that he can construct a totally real immersion of $S^3(1/16)$ into $S^6$.

**References**


