A PROBABILITY DISTRIBUTION ASSOCIATED WITH THE HURWITZ ZETA FUNCTION

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ABSTRACT. A new discrete probability distribution is defined in terms of the Hurwitz zeta function and its relationship to the Poisson distribution is demonstrated.

If we define the Hurwitz zeta function by

\[ \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \]

for \( a > 0 \) and \( s > 1 \), then it is easy to show that

\[ f(x; a) = a\zeta(x + 1, a + 1), \quad x = 1, 2, 3, \ldots, \]

is a probability density function (p.d.f) with factorial moment generating function

\[ G(t) = a \sum_{n=1}^{\infty} \left( \frac{1}{n + a - t} - \frac{1}{n + a} \right) \]

for \( |t| < 1 + a \) and factorial moments

\[ G^{(m)}(1) = am! \zeta(m + 1, a). \]

It will now be shown how \( f(x; a) \) arises in a natural way from the Poisson distribution. It is known [1, 13.12] that

\[ f(x; a) = g^*(x; y)h(y; a)dy, \]

where

\[ g^*(x; y) = \frac{g(x; y)}{1 - g(0; y)}, \quad x = 1, 2, 3, \ldots, \]

is the p.d.f. for a truncated Poisson distribution. Thus, by the same argument used in [2, 5.13B], \( f(x; a) \) is the p.d.f. for a compound truncated Poisson distribution where the parameter in the truncated Poisson distribution has an exponential distribution with mean \( 1/a \).

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Now let $a$ and $b$ be positive and replace $h(y; a)$ by the p.d.f. for the gamma distribution defined by

$$h(y; a, b) = \frac{a^b}{\Gamma(b)} y^{b-1} e^{-ay}, \quad 0 < y < \infty.$$ 

Then it follows as before that the compound truncated Poisson distribution, where the truncated Poisson distribution is the same as above but whose parameter now has a gamma distribution, will have the p.d.f.

$$f(x; a, b) = a^b (b + 1) \cdots (b + x - 1) \zeta(x + b, a + 1)/x!, \quad x = 1, 2, 3, \ldots,$$

and it can be shown that the factorial moments for this distribution are

$$G(m)(1) = a^b (b + 1) \cdots (b + m - 1) \zeta(m + b, a).$$

The corresponding problem for the compound Poisson distribution is considered in [2, 5.13B].

If $b/a = c = \text{constant}$ and $x$ is a positive integer, then

$$\zeta(x + b, a + 1) = \left( \frac{c}{b + c} \right)^{x + b} \sum_{n=0}^{\infty} \left( 1 + \frac{cn}{b + c} \right)^{-x - b}$$

and

$$f(x; a, b) = c^x \left( \frac{b}{b + c} \right)^b \left[ \frac{(b + 1) \cdots (b + x - 1)}{(b + c)^x} \right] \sum_{n=0}^{\infty} \left( 1 + \frac{cn}{b + c} \right)^{-x - b}.$$

Hence

$$\lim_{b \to \infty} f(x; a, b) = \frac{c^x e^{-c}}{x!} \sum_{n=0}^{\infty} e^{-cn} = \frac{c^x e^{-c}}{x!(1 - e^{-c})}.$$

Thus the truncated Poisson distribution with parameter $b/a$ provides a simple approximation for $f(x; a, b)$ when $a$ and $b$ are large.

For $t < \log(1 + a)$, the moment generating function is given by

$$M_X(t) = a^b \sum_{n=1}^{\infty} (n + a)^{-b} \sum_{x=1}^{\infty} \frac{b(b + 1) \cdots (b + x - 1)}{x!} \left[ \frac{e^t}{n + a} \right]^x$$

$$= a^b \sum_{n=1}^{\infty} (n + a)^{-b} \left( \frac{1 - e^t}{n + a} \right)^{-b}$$

$$= a^b \sum_{n=1}^{\infty} [(n + a - e^t)^{-b} - (n + a)^{-b}].$$

If $Y = (aX - b)/c$, where $c = \sqrt{b(a + 1)}$, then

$$M_Y(t) = \sum_{n=1}^{\infty} \left\{ \left[ \frac{a}{e^{t/c}(n + a - e^{at/c})} \right]^b - \frac{a}{e^{t/c}(n + a)} \right\}^b$$

and

$$\lim_{b \to \infty} M_Y(t) = \lim_{b \to \infty} \left[ \frac{a}{e^{t/c}(1 + a - e^{at/c})} \right]^b$$

$$= \lim_{b \to \infty} \left[ (1 + t/c + t^2/2c^2 + \cdots)(1 - t/c - at^2/2c^2 - \cdots) \right]^b$$

$$= \lim_{b \to \infty} \left[ 1 - t^2/2b + o(1/b) \right]^b = e^{t^2/2}.$$

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Hence $Y$ has an approximate standard normal distribution when $b$ is large.

If $Z = 2aX$, then

$$M_Z(t) = \sum_{n=1}^{\infty} \left[ \left( \frac{a}{n + a - e^{2at}} \right)^b - \left( \frac{a}{n + a} \right)^b \right]$$

and

$$\lim_{a \to 0} M_Z(t) = \lim_{a \to 0} \left( \frac{a}{1 + a - e^{2at}} \right)^b = (1 - 2t)^{-b}.$$

Thus $Z$ has an approximate chi-square distribution with $2b$ degrees of freedom when $a$ is small.

When $x = 1$, we have

$$f(1; a, b) = a^b b \zeta(b + 1, a + 1) = a^b b \sum_{n=1}^{\infty} (n + a)^{-b-1}$$

$$\geq a^b b \int_{1}^{\infty} (u + a)^{-b-1} du = \left( \frac{a}{a + 1} \right)^b.$$

Hence $X$ has a distribution which is approximately degenerate at one when $a$ is large or $b$ is small.

Finally, it has been observed by the referee that there are two other discrete distributions [3, 10.3] that are related to the Riemann zeta function $\zeta(s, 1)$.

REFERENCES

