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A PROBABILITY DISTRIBUTION ASSOCIATED WITH THE HURWITZ ZETA FUNCTION

R. L. DUNCAN

ABSTRACT. A new discrete probability distribution is defined in terms of the Hurwitz zeta function and its relationship to the Poisson distribution is demonstrated.

If we define the Hurwitz zeta function by

$$\varsigma(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

for $a > 0$ and $s > 1$, then it is easy to show that

$$f(x;a) = a\varsigma(x+1, a+1), \qquad x = 1, 2, 3, \ldots,$$

is a probability density function (p.d.f) with factorial moment generating function

$$G(t) = a \sum_{n=1}^{\infty} \left(\frac{1}{n+a-t} - \frac{1}{n+a} \right)$$

for $|t| < 1 + a$ and factorial moments

$$G^{(m)}(1) = am!\varsigma(m+1, a).$$

It will now be shown how $f(x;a)$ arises in a natural way from the Poisson distribution. It is known [**1**, 13.12] that

$$\varsigma(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{y^{s-1}e^{-ay}}{1-e^{-y}}\, dy.$$

If $g(x;y)$ and $h(y;a)$ are the p.d.f.s for the Poisson distribution with mean y and the exponential distribution with mean $1/a$, then

$$f(x;a) = \int_0^{\infty} g^*(x;y)h(y;a)\, dy,$$

where

$$g^*(x;y) = \frac{g(x;y)}{1-g(0;y)}, \qquad x = 1, 2, 3, \ldots,$$

is the p.d.f. for a truncated Poisson distribution. Thus, by the same argument used in [**2**, 5.13B], $f(x;a)$ is the p.d.f. for a compound truncated Poisson distribution where the parameter in the truncated Poisson distribution has an exponential distribution with mean $1/a$.

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Now let a and b be positive and replace $h(y;a)$ by the p.d.f. for the gamma distribution defined by

$$h(y;a,b) = \frac{a^b}{\Gamma(b)} y^{b-1} e^{-ay}, \qquad 0 < y < \infty.$$

Then it follows as before that the compound truncated Poisson distribution, where the truncated Poisson distribution is the same as above but whose parameter now has a gamma distribution, will have the p.d.f.

$$f(x;a,b) = a^b b(b+1)\cdots(b+x-1)\varsigma(x+b,a+1)/x!, \qquad x = 1,2,3,\ldots,$$

and it can be shown that the factorial moments for this distribution are

$$G^{(m)}(1) = a^b b(b+1)\cdots(b+m-1)\varsigma(m+b,a).$$

The corresponding problem for the compound Poisson distribution is considered in [**2**, 5.13B].

If $b/a = c =$ constant and x is a positive integer, then

$$\varsigma(x+b,a+1) = \left(\frac{c}{b+c}\right)^{x+b} \sum_{n=0}^{\infty} \left(1 + \frac{cn}{b+c}\right)^{-x-b}$$

and

$$f(x;a,b) = \frac{c^x}{x!}\left(\frac{b}{b+c}\right)^b \left[\frac{b(b+1)\cdots(b+x-1)}{(b+c)^x}\right] \sum_{n=0}^{\infty} \left(1 + \frac{cn}{b+c}\right)^{-x-b}.$$

Hence

$$\lim_{b\to\infty} f(x;a,b) = \frac{c^x e^{-c}}{x!} \sum_{n=0}^{\infty} e^{-cn} = \frac{c^x e^{-c}}{x!(1-e^{-c})}.$$

Thus the truncated Poisson distribution with parameter b/a provides a simple approximation for $f(x;a,b)$ when a and b are large.

For $t < \log(1+a)$, the moment generating function is given by

$$\begin{aligned} M_X(t) &= a^b \sum_{n=1}^{\infty} (n+a)^{-b} \sum_{x=1}^{\infty} \frac{b(b+1)\cdots(b+x-1)}{x!} \left[\frac{e^t}{n+a}\right]^x \\ &= a^b \sum_{n=1}^{\infty} (n+a)^{-b} \left[\left(1 - \frac{e^t}{n+a}\right)^{-b} - 1\right] \\ &= a^b \sum_{n=1}^{\infty} \left[(n+a-e^t)^{-b} - (n+a)^{-b}\right]. \end{aligned}$$

If $Y = (aX - b)/c$, where $c = \sqrt{b(a+1)}$, then

$$M_Y(t) = \sum_{n=1}^{\infty} \left\{ \left[\frac{a}{e^{t/c}(n+a-e^{at/c})}\right]^b - \left[\frac{a}{e^{t/c}(n+a)}\right]^b \right\}$$

and

$$\begin{aligned} \lim_{b\to\infty} M_Y(t) &= \lim_{b\to\infty} \left[a/e^{t/c}(1+a-e^{at/c})\right]^b \\ &= \lim_{b\to\infty} [(1+t/c+t^2/2c^2+\cdots)(1-t/c-at^2/2c^2-\cdots)]^{-b} \\ &= \lim_{b\to\infty} [1 - t^2/2b + o(1/b)]^{-b} = e^{t^2/2}. \end{aligned}$$

Hence Y has an approximate standard normal distribution when b is large.

If $Z = 2aX$, then

$$M_Z(t) = \sum_{n=1}^{\infty} \left[\left(\frac{a}{n + a - e^{2at}} \right)^b - \left(\frac{a}{n+a} \right)^b \right]$$

and

$$\lim_{a \to 0} M_Z(t) = \lim_{a \to 0} \left(\frac{a}{1 + a - e^{2at}} \right)^b = (1 - 2t)^{-b}.$$

Thus Z has an approximate chi-square distribution with $2b$ degrees of freedom when a is small.

When $x = 1$, we have

$$\begin{aligned} f(1; a, b) &= a^b b \varsigma(b+1, a+1) = a^b b \sum_{n=1}^{\infty} (n+a)^{-b-1} \\ &\geq a^b b \int_1^{\infty} (u+a)^{-b-1}\, du = \left(\frac{a}{a+1} \right)^b. \end{aligned}$$

Hence X has a distribution which is approximately degenerate at one when a is large or b is small.

Finally, it has been observed by the referee that there are two other discrete distributions [**3**, 10.3] that are related to the Riemann zeta function $\varsigma(s, 1)$.

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GRADUATE STUDIES AND CONTINUING EDUCATION, PENNSYLVANIA STATE UNIVERSITY, KING OF PRUSSIA, PENNSYLVANIA 19406