A PROBABILITY DISTRIBUTION ASSOCIATED WITH THE HURWITZ ZETA FUNCTION

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ABSTRACT. A new discrete probability distribution is defined in terms of the Hurwitz zeta function and its relationship to the Poisson distribution is demonstrated.

If we define the Hurwitz zeta function by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}$$

for $a > 0$ and $s > 1$, then it is easy to show that

$$f(x; a) = a\zeta(x + 1, a + 1), \quad x = 1, 2, 3, \ldots,$$

is a probability density function (p.d.f) with factorial moment generating function

$$G(t) = a \sum_{n=1}^{\infty} \left( \frac{1}{n + a - t} - \frac{1}{n + a} \right)$$

for $|t| < 1 + a$ and factorial moments

$$G^{(m)}(1) = am!\zeta(m + 1, a).$$

It will now be shown how $f(x; a)$ arises in a natural way from the Poisson distribution. It is known [1, 13.12] that

$$f(x; a) = \int_{0}^{\infty} g^*(x; y)h(y; a) dy,$$

where

$$g^*(x; y) = \frac{g(x; y)}{1 - g(0; y)}, \quad x = 1, 2, 3, \ldots,$$

is the p.d.f. for a truncated Poisson distribution. Thus, by the same argument used in [2, 5.13B], $f(x; a)$ is the p.d.f. for a compound truncated Poisson distribution where the parameter in the truncated Poisson distribution has an exponential distribution with mean $1/a$.

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Now let \( a \) and \( b \) be positive and replace \( h(y; a) \) by the p.d.f. for the gamma distribution defined by

\[
h(y; a, b) = \frac{a^b y^{b-1} e^{-ay}}{\Gamma(b)}, \quad 0 < y < \infty.
\]

Then it follows as before that the compound truncated Poisson distribution, where the truncated Poisson distribution is the same as above but whose parameter now has a gamma distribution, will have the p.d.f.

\[
f(x; a, b) = a^b b(b + 1) \cdots (b + x - 1) \zeta(x + b, a + 1)/x!, \quad x = 1, 2, 3, \ldots,
\]

and it can be shown that the factorial moments for this distribution are

\[
G^{(m)}(1) = a^b b(b + 1) \cdots (b + m - 1) \zeta(m + b, a).
\]

The corresponding problem for the compound Poisson distribution is considered in [2, 5.13B].

If \( b/a = c = \text{constant} \) and \( x \) is a positive integer, then

\[
\zeta(x + b, a + 1) = \left( \frac{c}{b + c} \right)^{x+b} \sum_{n=0}^{\infty} \left( 1 + \frac{cn}{b + c} \right)^{-x-b}.
\]

and

\[
f(x; a, b) = \frac{c^x}{x!} \left( \frac{b}{b + c} \right)^b \left[ \frac{b(b + 1) \cdots (b + x - 1)}{(b + c)^x} \right] \sum_{n=0}^{\infty} \left( 1 + \frac{cn}{b + c} \right)^{-x-b}.
\]

Hence

\[
\lim_{b \to \infty} f(x; a, b) = \frac{c^x e^{-c}}{x!} \sum_{n=0}^{\infty} e^{-cn} = \frac{c^x e^{-c}}{x!(1 - e^{-c})}.
\]

Thus the truncated Poisson distribution with parameter \( b/a \) provides a simple approximation for \( f(x; a, b) \) when \( a \) and \( b \) are large.

For \( t < \log(1 + a) \), the moment generating function is given by

\[
M_X(t) = a^b \sum_{n=1}^{\infty} (n + a)^{-b} \sum_{x=1}^{\infty} \frac{b(b + 1) \cdots (b + x - 1)}{x!} \left[ \frac{e^t}{n + a} \right]^x
\]

\[
= a^b \sum_{n=1}^{\infty} (n + a)^{-b} \left[ \left( 1 - \frac{e^t}{n + a} \right)^{-b} - 1 \right]
\]

\[
= a^b \sum_{n=1}^{\infty} [(n + a - e^t)^{-b} - (n + a)^{-b}].
\]

If \( Y = (aX - b)/c \), where \( c = \sqrt{b(a + 1)} \), then

\[
M_Y(t) = \sum_{n=1}^{\infty} \left\{ \left[ \frac{a}{e^{t/c}(n + a - e^{at/c})} \right]^b - \frac{a}{e^{t/c}(n + a)} \right\}^b
\]

and

\[
\lim_{b \to \infty} M_Y(t) = \lim_{b \to \infty} \left[ a/e^{t/c}(1 + a - e^{at/c}) \right]^b
\]

\[
= \lim_{b \to \infty} \left[ (1 + t/c + t^2/2c^2 + \cdots)(1 - t/c - at^2/2c^2 - \cdots) \right]^{-b}
\]

\[
= \lim_{b \to \infty} \left[ 1 - t^2/2b + o(1/b) \right]^{-b} = e^{t^2/2}.
\]
Hence $Y$ has an approximate standard normal distribution when $b$ is large.

If $Z = 2aX$, then

$$M_Z(t) = \sum_{n=1}^{\infty} \left[ \left( \frac{a}{n + a - e^{2at}} \right)^b - \left( \frac{a}{n + a} \right)^b \right]$$

and

$$\lim_{a \to 0} M_Z(t) = \lim_{a \to 0} \left( \frac{a}{1 + a - e^{2at}} \right)^b = (1 - 2t)^{-b}.$$

Thus $Z$ has an approximate chi-square distribution with $2b$ degrees of freedom when $a$ is small.

When $x = 1$, we have

$$f(1; a, b) = a^b b \zeta(b + 1, a + 1) = a^b b \sum_{n=1}^{\infty} (n + a)^{-b-1}$$

$$\geq a^b b \int_{1}^{\infty} (u + a)^{-b-1} du = \left( \frac{a}{a + 1} \right)^b.$$

Hence $X$ has a distribution which is approximately degenerate at one when $a$ is large or $b$ is small.

Finally, it has been observed by the referee that there are two other discrete distributions [3, 10.3] that are related to the Riemann zeta function $\zeta(s, 1)$.

REFERENCES