LOCAL CUT-POINTS IN CONTINUOUS IMAGES OF COMPACT ORDERED SPACES

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Abstract. We prove that if a continuum $X$ is continuous image of a compact ordered space and if $X$ is not locally separable at a point $x$, then $x$ lies in the closure of the set of all local cut-points of $X$.

A continuum is a compact connected Hausdorff space. An ordered space is a totally ordered set endowed with the topology generated by open intervals. It is known from L. B. Treybig [4] that if a nonmetrizable continuum $X$ is a continuous image of a compact ordered space, then some subset of less than three points separates $X$. In this paper we prove the following theorem by modifying Treybig’s method, and give a corollary.

Theorem. If a continuum $X$ is a continuous image of a compact ordered space and if $X$ is not locally separable at $x_0$, then for any neighborhood $U$ of $x_0$ there exists a nonempty open set $W$ such that $\overline{W} \subset U$ and the boundary $\text{Bd}(W)$ consists of at most two points.

A space $X$ is said to be locally separable at $x$ if there is a separable neighborhood of $x$. A point $x$ of $X$ is a local cut-point of $X$ if there is a connected open neighborhood $U$ of $x$ such that $U - \{x\}$ is not connected. We will denote the set of all local cut-points of $X$ by $L(X)$. If $W$ is a nonempty open subset of a continuum $X$ such that its closure $\overline{W}$ is not $X$, then its boundary $\text{Bd}(W)$ separates $X$. In addition, if $\text{Bd}(W)$ is finite, $\text{Bd}(W)$ contains at least one local cut-point. Hence, the theorem implies:

Corollary. Let $X$ be a continuum which is a continuous image of a compact ordered space. If $X$ is not locally separable at $x_0$, then $x_0$ is in the closure of $L(X)$.

In the preceding corollary, even if $X$ is locally connected, we can assert neither that $x_0$ is always an accumulation point of $L(X)$, nor that $x_0$ is a local cut-point or an end-point of $X$. Counterexamples for these will be given in §3.

1. We prepare an essential part of Treybig’s method used in [4 and 5] in a slightly modified form as follows:

Lemma. Let $K$ be a nonseparable compact ordered space, and let $f$ be a continuous map from $K$ onto a Hausdorff space. Given a countable subset $L_0$ of $K$, $K$ can be
written as the union of mutually disjoint nonempty subsets,
\[ K = L \cup \left( \bigcup \mathcal{G}_\lambda \right). \]

such that the following conditions are satisfied:

1. \( L \cap G_\lambda = \emptyset, \ G_\lambda \cap G_\mu = \emptyset \) (if \( \lambda \neq \mu \)),
2. \( L \) is a separable closed set containing \( L_0 \),
3. each \( G_\lambda \) is an open set of the form \((a_\lambda, b_\lambda)\),
4. if \( f(G_\lambda) \cap f(G_\mu) \neq \emptyset \), then \( f(a_\lambda), f(b_\lambda) = \{ f(a_\mu), f(b_\mu) \} \), and
5. \( f(L) \cap f(G_\lambda) \subset \{ f(a_\lambda), f(b_\lambda) \} \).

**Proof.** Let \( L_0 = \{ c_1, c_2, c_3, \ldots \} \). We construct a sequence of finite subsets \( L_1, L_2, L_3, \ldots \) of \( K \) inductively as follows. Let \( a_0, a_2 \) be the minimum and the maximum of \( K \), respectively. Choose \( a_1 \) so that \( a_0 < a_1 < a_2 \), and define \( L_1 = [a_0, a_1, a_2) \). Suppose that \( L_n = [b_0, b_1, \ldots, b_p] \) is defined, where \( b_0 < b_1 < \cdots < b_p \). Put \( I_i = [b_{i-1}, b_i] \) for \( i = 1, 2, \ldots, p \). If \( i \neq j \), we define

\[ A_{ij} = \{ (\xi, \eta) \in I_i \times I_j : f(\xi) = f(\eta) \}. \]

If \( A_{ij} \) is nonempty, there exists a subset \( \{(\alpha, \alpha'), (\beta, \beta'), (\gamma, \gamma'), (\delta, \delta')\} \) of \( A_{ij} \) such that

\[ \alpha < \xi < \beta, \quad \gamma < \eta < \delta \quad \text{for all} \ (\xi, \eta) \in A_{ij}. \]

We set \( B_{ij} = \{ \alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta' \} \). If \( A_{ij} \) is empty we set \( B_{ij} = \emptyset \). Define

\[ L_{n+1} = \bigcup_{i \neq j} B_{ij} \cup \{ c_1, c_2, \ldots, c_n \}, \]

and let \( L = \bigcup_n L_n \). Then \( L \) is a separable closed set. Since \( K \) is nonseparable, \( K - L \) is a nonempty open subset of \( K \). \( K - L \) can be written as the union of mutually disjoint maximal convex open subsets \( G_\lambda \). Since the minimum and the maximum of \( K \) belong to \( L \), each \( G_\lambda \) is of the form \((a_\lambda, b_\lambda)\), where \( a_\lambda, b_\lambda \in L \).

To prove (4), suppose that \( f(c) = f(d) \) for \( c \in G_\lambda \) and \( d \in G_\mu \). Also suppose that

\[ f(a_\lambda) \notin \{ f(a_\mu), f(b_\mu) \}. \]

Then there exist convex open neighborhoods \( U_\lambda, U_\mu, V_\mu \) of \( a_\lambda, a_\mu, b_\mu \), respectively, such that

\[ f(U_\lambda) \cap f(U_\mu) = \emptyset \quad \text{and} \quad f(U_\lambda) \cap f(V_\mu) = \emptyset. \]

By the definition of \( L \), for some integer \( n \) the set \( L_n = \{ b_0, b_1, \ldots, b_p \} \) contains elements \( b_i, b_j \) such that

\[(a_\lambda, b_\lambda) \subset [b_{i-1}, b_i], \quad (a_\mu, b_\mu) \subset [b_{j-1}, b_j], \]

\[ b_{i-1} \in U_\lambda, \quad b_{j-1} \in U_\mu, \quad b_j \in V_\mu. \]

Since \( f(c) = f(d) \), by construction, the set \( L_{n+1} \) contains elements \( \alpha, \alpha' \in B_{ij} \) such that

\[ b_{i-1} \leq \alpha \leq c, \quad b_{j-1} \leq \alpha' \leq b_j, \quad \text{and} \quad f(\alpha) = f(\alpha'). \]

If \( a_\lambda < \alpha \), then we would have \( a_\lambda < \alpha < b_\lambda \) and hence \( \alpha \in G_\lambda \cap L_{n+1} \) (contradiction). If \( \alpha \leq a_\lambda \), then \( \alpha \in U_\lambda \). So \( \alpha' \) could not lie in \( U_\mu \cup V_\mu \), and hence \( a_\mu < \alpha' < b_\mu \) (contradiction). A similar argument shows that \( f(b_\lambda) \notin \{ f(a_\mu), f(b_\mu) \} \) and \{ \( f(a_\mu), f(b_\mu) \} \subset \{ f(a_\lambda), f(b_\lambda) \} \).
To prove (5), suppose that \( f(c) = f(d) \) for \( c \in L \) and \( d \in G_\lambda \), and that \( f(c) \notin \{ f(a_\lambda), f(b_\lambda) \} \). Choose convex neighborhoods \( W, U_\lambda, V_\lambda \) of \( c, a_\lambda, b_\lambda \), respectively so that

\[
f(W) \cap f(U_\lambda) = \emptyset, \quad f(W) \cap f(V_\lambda) = \emptyset.
\]

For some integer \( n \), the set \( L_n = \{ b_0, b_1, \ldots, b_p \} \) contains such elements \( b_i, b_j \) that satisfy:

\[
b_{i-1} \leq c \leq b_i, \quad b_{j-1} \leq a_\lambda < d < b_\lambda \leq b_j,
\]

\[
b_{j-1} \in U_\lambda, \quad b_j \in V_\lambda, \quad \text{and} \quad \{ b_{j-1}, b_j \} \cap W \neq \emptyset.
\]

Then the set \( L_{n+1} \) contains elements \( \alpha, \alpha', \beta, \beta' \) such that:

\[
b_{i-1} \leq \alpha \leq c \leq \beta \leq b_i, \quad \alpha', \beta' \in [b_{j-1}, b_j],
\]

\[
f(\alpha) = f(\alpha') \quad \text{and} \quad f(\beta) = f(\beta').
\]

If \( b_{i-1} \in W \), then \( \alpha \in W \) and so \( \alpha' \notin U_\lambda \cup V_\lambda \), whence \( \alpha' \in (a_\lambda, b_\lambda) \). This is a contradiction. Similarly, if \( b_j \in W \), then we would have \( \beta' \in (a_\lambda, b_\lambda) \).

2. Proof of the theorem. We assume \( U \neq X \). Since \( X \) is normal, we can find a sequence of open sets \( U_1, U_2, U_3, \ldots \) such that

\[
x_0 \in U_n \subset \overline{U}_n \subset U_{n+1} \subset \overline{U}_{n+1} \subset U.
\]

Let \( S = X - (\cup \{ U_n, n = 1, 2, 3, \ldots \}) \); then \( S \) is a nonempty closed set and \( x_0 \notin S \). We define an equivalence relation \( \sim \) on \( X \) by setting \( x \sim y \) if and only if \( x = y \) or \( \{ x, y \} \subset S \). Since \( \sim \) is a closed relation, the quotient space \( X_1 = X/\sim \) is Hausdorff. Note that the natural projection \( \phi: X \to X_1 \) is a local homeomorphism on \( X - S \), and hence, \( X_1 \) is not separable. Let \( V_n = \phi(X - \overline{U}_n) \). Then \( \{ V_n \}_{n=1}^{\infty} \) forms a countable basis of neighborhoods at point \( s = \phi(S) \) in \( X_1 \).

As well as \( X, X_1 \) is a continuous image of compact ordered space. Let \( K \) be a compact ordered space, and \( f \) a continuous surjection from \( K \) onto \( X_1 \). By [3, Lemma 4] we can assume that (i) if \( K_1 \) is a closed proper subset of \( K \), then \( f(K_1) \neq X_1 \), and (ii) if \( a, b \) are different elements of \( K \) with \( f(a) = f(b) \), then there is a point \( c \) between \( a \) and \( b \) such that \( f(c) \neq f(a) \).

For each positive integer \( n \), there is a finite cover \( \Omega_n \) of \( f^{-1}(s) \) such that each member of \( \Omega_n \) is of the form \( (a_i, b_i) \), or \( (a_i, \infty) \), or \( (-\infty, b_i) \), and is contained in \( f^{-1}(V_n) \). Let \( L_n \) denote the set of all end-points \( a_i, b_j \) of members of \( \Omega_n \), and let \( L_0 = \bigcup_n L_n \). We remark that if two points \( a, b \) of \( K \) are not consecutive and \( f(a) = s \), then there is a point of \( L_0 \) between \( a \) and \( b \). In fact, choose a point \( c \) between \( a \) and \( b \). By (ii), we can assume \( f(c) \neq s \). There is an integer \( n \) such that \( f(c) \notin V_n \). Then one of the end-points of a member of \( \Omega_n \) which contains \( a \) must lie between \( a \) and \( b \).

If necessary, adding a point of \( f^{-1}(s) \) to \( L_0 \), we can assume \( s \in f(L_0) \). Noting that \( K \) is nonseparable, we apply our lemma for the map \( f: K \to X_1 \) and the countable subset \( L_0 \) of \( K \) to obtain a decomposition of \( K \) into mutually disjoint nonempty subsets with properties (1)-(5) as in the lemma:
We choose one \( G_\lambda \) and denote it by \( G = (a, b) \). Let \( H^* \) be the collection of all \( G_\mu \) such that \( \{ f(a_\mu), f(b_\mu) \} = \{ f(a), f(b) \} \), and let \( H = \bigcup H^* \) and \( M = \{ \mu : G_\mu \in H^* \} \).

Let \( W_1 = X_1 - f(K - H) \). \( W_1 \) is open. \( W_1 \) is nonempty, since otherwise we would have \( X_1 = f(K - H) \) where \( K - H \) is a proper closed subset of \( K \), contradicting assumption (i) on map \( f \). Since \( W_1 \subset f(H) \subset f(H) \), it follows that \( \overline{W_1} \subset f(H) \). Now let \( c \) be a boundary point of the open set \( H \), and let \( V \) be an arbitrary convex neighborhood of \( c \). Since \( V \) meets \( H \), \( V \) meets some member \( G_\mu \) of \( H^* \). But \( c \) is not in \( G_\mu \). Then one of \( a_\mu \) or \( b_\mu \) must lie in \( V \). Thus \( c \) is in the closure of \( \{ a_\mu, b_\mu : \mu \in M \} \) and hence in the closed set \( f^{-1}(a) \cup f^{-1}(b) \). Therefore,

\[
\overline{H} \subset H \cup f^{-1}(a) \cup f^{-1}(b),
\]

and

\[
\overline{W_1} \subset f(H) \subset f(H) \cup \{ f(a), f(b) \}.
\]

It follows that

\[
\text{Bd}(W_1) = \overline{W_1} \cap X_1 - W_1 \subset (f(H) \cup \{ f(a), f(b) \}) \cap f(K - H)
= \{ f(a), f(b) \} \quad \text{(by properties (4) and (5)).}
\]

Note that \( s \notin \{ f(a), f(b) \} \), since if for example \( f(a) = s \), then as remarked previously, some element of \( L_0 \) would lie in \( G = (a, b) \), contradicting \( G \cap L = \emptyset \). Thus \( s \notin \overline{W_1} \). Finally, since \( \phi : X \to X_1 \) is locally homeomorphic on \( X - S \), and \( s = \phi(S) \notin \overline{W_1} \), the open set \( W = \phi^{-1}(W_1) \) of \( X \) satisfies the required properties in the theorem.

### 3. Examples

As mentioned in the introduction, we give two examples to complement our corollary.

**Example 1.** Let \( \Lambda \) be an uncountable set. For each \( \lambda \in \Lambda \), let \( D_\lambda \) denote the unit closed disk with center \( o \) in the complex number plane, and let \( Y \) denote the product of all \( D_\lambda \). We regard each \( D_\lambda \) as a subset of \( Y \) by identifying a point \( z \in D_\lambda \) with the point \( (z_\mu) \) of \( Y \) defined by \( z_\mu = z \) if \( \mu = \lambda \), and \( z_\mu = o \) if \( \mu \neq \lambda \). Then all subsets \( D_\lambda \) have the common center \( O \). By the definition of the product topology, we note that every neighborhood of \( O \) contains all \( D_\lambda \) except for finitely many \( \lambda \). Define \( X = \bigcup D_\lambda \) and endow \( X \) with the relative topology from \( Y \). Then space \( X \) has the following properties:

1. \( X \) is a locally connected continuum,
2. \( X \) is not locally separable at \( O \), and is locally separable at any other points,
3. \( O \) is a cut-point of \( X \) and it is a unique local cut-point,
4. \( X \) is two dimensional at every point, and
5. \( X \) is continuous image of an ordered continuum.

Properties (1)–(4) are easily verified. Property (5) follows from a theorem of Cornette [1] that states that a locally connected continuum is a continuous image of an ordered continuum if and only if its every cyclic element is so. Note that the cyclic elements of \( X \) are the \( D_\lambda \) which are locally connected metric continuum and hence are continuous images of the real interval \([0, 1]\) by the classical theorem of Hahn-Mazurkiewicz.
EXAMPLE 2. Let $I$ denote the interval $[0, 1]$ of the real numbers, and $L$ the "long
interval" $[0, \Omega]$, which is a nonseparable ordered continuum. We define a subspace $Y$ of $L \times I$ by

$$Y = A_0 \cup \left( \bigcup_n A_n \right) \cup B_0 \cup B_\Omega,$$

where

$$A_0 = \left\{ (\lambda, 0) : \lambda \in L \right\},$$
$$A_n = \left\{ (\lambda, 1/n) : \lambda \in L \right\}, \quad \text{where } n = 1, 2, 3, \ldots,$$
$$B_0 = \left\{ (0, t) : t \in I \right\}, \quad B_\Omega = \left\{ (\Omega, t) : t \in I \right\}.$$

Define an equivalence relation $\sim$ on $Y$ by setting $y \sim z$ if and only if $y = z$ or $\{y, z\} \subset A_0$. Let $X$ denote the quotient space $Y/\sim$, and $a_0$ the point which is the image of $A_0$ under the natural projection. Then $X$ satisfies the following properties:

1. $X$ is a continuum,
2. $X$ is netlike (that means, any two different points of $X$ can be separated by a
finite subset of $X$),
3. $X$ is not locally separable at $a_0$,
4. $a_0$ is neither a local cut-point nor an end-point of $X$, and
5. $X$ is a continuous image of an ordered continuum.

Properties (1)–(4) are easily verified. Property (5) follows from (1) and (2) (see [2
or 6]).

REFERENCES