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LOCAL CUT-POINTS IN CONTINUOUS IMAGES OF COMPACT ORDERED SPACES

RYUJI MAEHARA

ABSTRACT. We prove that if a continuum X is continuous image of a compact ordered space and if X is not locally separable at a point x, then x lies in the closure of the set of all local cut-points of X.

A *continuum* is a compact connected Hausdorff space. An *ordered space* is a totally ordered set endowed with the topology generated by open intervals. It is known from L. B. Treybig [**4**] that if a nonmetrizable continuum X is a continuous image of a compact ordered space, then some subset of less than three points separates X. In this paper we prove the following theorem by modifying Treybig's method, and give a corollary.

THEOREM. *If a continuum X is a continuous image of a compact ordered space and if X is not locally separable at x_0, then for any neighborhood U of x_0 there exists a nonempty open set W such that $\overline{W} \subset U$ and the boundary* $\mathrm{Bd}(W)$ *consists of at most two points.*

A space X is said to be *locally separable* at x if there is a separable neighborhood of x. A point x of X is a *local cut-point* of X if there is a connected open neighborhood U of x such that $U - \{x\}$ is not connected. We will denote the set of all local cut-points of X by $L(X)$. If W is a nonempty open subset of a continuum X such that its closure $\overline{W}$ is not X, then its boundary $\mathrm{Bd}(W)$ separates X. In addition, if $\mathrm{Bd}(W)$ is finite, $\mathrm{Bd}(W)$ contains at least one local cut-point. Hence, the theorem implies:

COROLLARY. *Let X be a continuum which is a continuous image of a compact ordered space. If X is not locally separable at x_0, then x_0 is in the closure of $L(X)$.*

In the preceding corollary, even if X is locally connected, we can assert neither that x_0 is always an accumulation point of $L(X)$, nor that x_0 is a local cut-point or an end-point of X. Counterexamples for these will be given in §3.

1. We prepare an essential part of Treybig's method used in [**4** and **5**] in a slightly modified form as follows:

LEMMA. *Let K be a nonseparable compact ordered space, and let f be a continuous map from K onto a Hausdorff space. Given a countable subset L_0 of K, K can be*

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written as the union of mutually disjoint nonempty subsets,

$$K = L \cup \left(\bigcup_{\lambda} G_\lambda\right),$$

such that the following conditions are satisfied:

(1) $L \cap G_\lambda = \varnothing$, $G_\lambda \cap G_\mu = \varnothing$ (*if* $\lambda \neq \mu$),

(2) L *is a separable closed set containing* L_0,

(3) *each* G_λ *is an open set of the form* (a_λ, b_λ),

(4) *if* $f(G_\lambda) \cap f(G_\mu) \neq \varnothing$, *then* $\{f(a_\lambda), f(b_\lambda)\} = \{f(a_\mu), f(b_\lambda)\}$, *and*

(5) $f(L) \cap f(G_\lambda) \subset \{f(a_\lambda), f(b_\lambda)\}$.

PROOF. Let $L_0 = \{c_1, c_2, c_3, \dots\}$. We construct a sequence of finite subsets L_1, L_2, $L_3, \dots$ of K inductively as follows. Let a_0, a_2 be the minimum and the maximum of K, respectively. Choose a_1 so that $a_0 < a_1 < a_2$, and define $L_1 = \{a_0, a_1, a_2\}$. Suppose that $L_n = \{b_0, b_1, \dots, b_p\}$ is defined, where $b_0 < b_1 < \cdots < b_p$. Put $I_i = [b_{i-1}, b_i]$ for $i = 1, 2, \dots, p$. If $i \neq j$, we define

$$A_{ij} = \left\{(\xi, \eta) \in I_i \times I_j \colon f(\xi) = f(\eta)\right\}.$$

If A_{ij} is nonempty, there exists a subset $\{(\alpha, \alpha'), (\beta, \beta'), (\gamma', \gamma), (\delta', \delta)\}$ of A_{ij} such that

$$\alpha \leqslant \xi \leqslant \beta, \quad \gamma \leqslant \eta \leqslant \delta \quad \text{for all } (\xi, \eta) \in A_{ij}.$$

We set $B_{ij} = \{\alpha, \alpha', \beta, \beta', \gamma', \gamma, \delta', \delta\}$. If A_{ij} is empty we set $B_{ij} = \varnothing$. Define

$$L_{n+1} = L_n \cup \left(\bigcup_{i \neq j} B_{ij}\right) \cup \{c_1, c_2, \dots, c_n\},$$

and let $L = \overline{\bigcup_n L_n}$. Then L is a separable closed set. Since K is nonseparable, $K - L$ is a nonempty open subset of K. $K - L$ can be written as the union of mutually disjoint maximal convex open subsets G_λ. Since the minimum and the maximum of K belong to L, each G_λ is of the form (a_λ, b_λ), where $a_\lambda, b_\lambda \in L$.

To prove (4), suppose that $f(c) = f(d)$ for $c \in G_\lambda$ and $d \in G_\mu$. Also suppose that $f(a_\lambda) \notin \{f(a_\mu), f(b_\mu)\}$. Then there exist convex open neighborhoods U_λ, U_μ, V_μ of a_λ, a_μ, b_μ, respectively, such that

$$f(U_\lambda) \cap f(U_\mu) = \varnothing \quad \text{and} \quad f(U_\lambda) \cap f(V_\mu) = \varnothing.$$

By the definition of L, for some integer n the set $L_n = \{b_0, b_1, \dots, b_p\}$ contains elements b_i, b_j such that

$$(a_\lambda, b_\lambda) \subset [b_{i-1}, b_i], \qquad (a_\mu, b_\mu) \subset [b_{j-1}, b_j],$$
$$b_{i-1} \in U_\lambda, \qquad b_{j-1} \in U_\mu, \qquad b_j \in V_\mu.$$

Since $f(c) = f(d)$, by construction, the set L_{n+1} contains elements α, $\alpha' \in B_{ij}$ such that

$$b_{i-1} \leqslant \alpha \leqslant c, \quad b_{j-1} \leqslant \alpha' \leqslant b_j, \quad \text{and} \quad f(\alpha) = f(\alpha').$$

If $a_\lambda < \alpha$, then we would have $a_\lambda < \alpha < b_\lambda$ and hence $\alpha \in G_\lambda \cap L_{n+1}$ (contradiction). If $\alpha \leqslant a_\lambda$, then $\alpha \in U_\lambda$. So α' could not lie in $U_\mu \cup V_\mu$, and hence $a_\mu < \alpha' < b_\mu$ (contradiction). A similar argument shows that $f(b_\lambda) \in \{f(a_\mu), f(b_\mu)\}$ and $\{f(a_\mu), f(b_\mu)\} \subset \{f(a_\lambda), f(b_\lambda)\}$.

To prove (5), suppose that $f(c) = f(d)$ for $c \in L$ and $d \in G_\lambda$, and that $f(c) \notin \{f(a_\lambda), f(b_\lambda)\}$. Choose convex neighborhoods W, U_λ, V_λ of c, a_λ, b_λ, respectively so that

$$f(W) \cap f(U_\lambda) = \varnothing, \qquad f(W) \cap f(V_\lambda) = \varnothing.$$

For some integer n, the set $L_n = \{b_0, b_1, \dots, b_p\}$ contains such elements b_i, b_j that satisfy:

$$b_{i-1} \leqslant c \leqslant b_i, \qquad b_{j-1} \leqslant a_\lambda < d < b_\lambda \leqslant b_j,$$
$$b_{j-1} \in U_\lambda, \quad b_j \in V_\lambda, \quad \text{and} \quad \{b_{i-1}, b_i\} \cap W \neq \varnothing.$$

Then the set L_{n+1} contains elements α, α', β, β' such that

$$b_{i-1} \leqslant \alpha \leqslant c \leqslant \beta \leqslant b_i, \qquad \alpha', \beta' \in [b_{j-1}, b_j],$$
$$f(\alpha) = f(\alpha') \quad \text{and} \quad f(\beta) = f(\beta').$$

If $b_{i-1} \in W$, then $\alpha \in W$ and so $\alpha' \notin U_\lambda \cup V_\lambda$, whence $\alpha' \in (a_\lambda, b_\lambda)$. This is a contradiction. Similarly, if $b_i \in W$, then we would have $\beta' \in (a_\lambda, b_\lambda)$.

2. Proof of the theorem. We assume $U \neq X$. Since X is normal, we can find a sequence of open sets $U_1, U_2, U_3, \dots$ such that

$$x_0 \in U_n \subset \overline{U}_n \subset U_{n+1} \subset \overline{U}_{n+1} \subset U.$$

Let $S = X - (\bigcup\{U_n\colon n = 1, 2, 3, \dots\})$; then S is a nonempty closed set and $x_0 \notin S$. We define an equivalence relation $\sim$ on X by setting $x \sim y$ if and only if $x = y$ or $\{x, y\} \subset S$. Since $\sim$ is a closed relation, the quotient space $X_1 = X/\sim$ is Hausdorff. Note that the natural projection $\phi\colon X \to X_1$ is a local homeomorphism on $X - S$, and hence, X_1 is not separable. Let $V_n = \phi(X - \overline{U}_n)$. Then $\{V_n\}_{n=1}^\infty$ forms a countable basis of neighborhoods at point s $(= \phi(S))$ in X_1.

As well as X, X_1 is a continuous image of compact ordered space. Let K be a compact ordered space, and f a continuous surjection from K onto X_1. By [**3**, Lemma 4] we can assume that (i) if K_1 is a closed proper subset of K, then $f(K_1) \neq X_1$, and (ii) if a, b are different elements of K with $f(a) = f(b)$, then there is a point c between a and b such that $f(c) \neq f(a)$.

For each positive integer n, there is a finite cover Ω_n of $f^{-1}(s)$ such that each member of Ω_n is of the form (a_i, b_i), or (a_i, ∞), or $(-\infty, b_i)$, and is contained in $f^{-1}(V_n)$. Let L_n denote the set of all end-points a_i, b_i of members of Ω_n, and let $L_0 = \bigcup_n L_n$. We remark that if two points a, b of K are not consecutive and $f(a) = s$, then there is a point of L_0 between a and b. In fact, choose a point c between a and b. By (ii), we can assume $f(c) \neq s$. There is an integer n such that $f(c) \notin V_n$. Then one of the end-points of a member of Ω_n which contains a must lie between a and b.

If necessary, adding a point of $f^{-1}(s)$ to L_0, we can assume $s \in f(L_0)$. Noting that K is nonseparable, we apply our lemma for the map $f\colon K \to X_1$ and the countable subset L_0 of K to obtain a decomposition of K into mutually disjoint nonempty subsets with properties (1)–(5) as in the lemma:

$$K = L \cup \left(\bigcup_\lambda G_\lambda\right).$$

We choose one G_λ and denote it by $G = (a, b)$. Let H^* be the collection of all G_μ such that $\{f(a_\mu), f(b_\mu)\} = \{f(a), f(b)\}$, and let $H = \bigcup H^*$ and $M = \{\mu\colon G_\mu \in H^*\}$.

Let $W_1 = X_1 - f(K - H)$. W_1 is open. W_1 is nonempty, since otherwise we would have $X_1 = f(K - H)$ where $K - H$ is a proper closed subset of K, contradicting assumption (i) on map f. Since $W_1 \subset f(H) \subset f(\overline{H})$, it follows that $\overline{W_1} \subset f(\overline{H})$. Now let c be a boundary point of the open set H, and let V be an arbitrary convex neighborhood of c. Since V meets H, V meets some member G_μ of H^*. But c is not in G_μ. Then one of a_μ or b_μ must lie in V. Thus c is in the closure of $\{a_\mu, b_\mu\colon \mu \in M\}$ and hence in the closed set $f^{-1}f(a) \cup f^{-1}f(b)$. Therefore,

$$\overline{H} \subset H \cup f^{-1}f(a) \cup f^{-1}f(b),$$

and

$$\overline{W_1} \subset f(\overline{H}) \subset f(H) \cup \{f(a), f(b)\}.$$

It follows that

$$\begin{aligned}\mathrm{Bd}(W_1) &= \overline{W_1} \cap \overline{X_1 - W_1} \subset (f(H) \cup \{f(a), f(b)\}) \cap f(K - H)\\ &= \{f(a), f(b)\} \quad \text{(by properties (4) and (5))}.\end{aligned}$$

Note that $s \notin \{f(a), f(b)\}$, since if for example $f(a) = s$, then as remarked previously, some element of L_0 would lie in $G = (a, b)$, contradicting $G \cap L = \varnothing$. Thus $s \notin \overline{W_1}$. Finally, since $\phi\colon X \to X_1$ is locally homeomorphic on $X - S$, and $s = \phi(S) \notin \overline{W_1}$, the open set $W = \phi^{-1}(W_1)$ of X satisfies the required properties in the theorem.

3. Examples. As mentioned in the introduction, we give two examples to complement our corollary.

EXAMPLE 1. Let Λ be an uncountable set. For each $\lambda \in \Lambda$, let D_λ denote the unit closed disk with center o in the complex number plane, and let Y denote the product of all D_λ. We regard each D_λ as a subset of Y by identifying a point $z \in D_\lambda$ with the point (z_μ) of Y defined by $z_\mu = z$ if $\mu = \lambda$, and $z_\mu = o$ if $\mu \neq \lambda$. Then all subsets D_λ have the common center O. By the definition of the product topology, we note that every neighborhood of O contains all D_λ except for finitely many λ. Define $X = \bigcup_\lambda D_\lambda$ and endow X with the relative topology from Y. Then space X has the following properties:

(1) X is a locally connected continuum,

(2) X is not locally separable at O, and is locally separable at any other points,

(3) O is a cut-point of X and it is a unique local cut-point,

(4) X is two dimensional at every point, and

(5) X is continuous image of an ordered continuum.

Properties (1)–(4) are easily verified. Property (5) follows from a theorem of Cornette [**1**] that states that a locally connected continuum is a continuous image of an ordered continuum if and only if its every cyclic element is so. Note that the cyclic elements of X are the D_λ which are locally connected metric continuum and hence are continuous images of the real interval $[0, 1]$ by the classical theorem of Hahn-Mazurkiewicz.

EXAMPLE 2. Let I denote the interval $[0,1]$ of the real numbers, and L the "long interval" $[0,\Omega]$, which is a nonseparable ordered continuum. We define a subspace Y of $L \times I$ by

$$Y = A_0 \cup \left(\bigcup_n A_n\right) \cup B_0 \cup B_\Omega,$$

where

$$A_0 = \{(\lambda,0)\colon \lambda \in L\},$$
$$A_n = \{(\lambda,1/n)\colon \lambda \in L\}, \quad \text{where } n = 1,2,3,\ldots,$$
$$B_0 = \{(0,t)\colon t \in I\}, \qquad B_\Omega = \{(\Omega,t)\colon t \in I\}.$$

Define an equivalence relation $\sim$ on Y by setting $y \sim z$ if and only if $y = z$ or $\{y,z\} \subset A_0$. Let X denote the quotient space $Y/\sim$, and a_0 the point which is the image of A_0 under the natural projection. Then X satisfies the following properties:

(1) X is a continuum,

(2) X is netlike (that means, any two different points of X can be separated by a finite subset of X),

(3) X is not locally separable at a_0,

(4) a_0 is neither a local cut-point nor an end-point of X, and

(5) X is a continuous image of an ordered continuum.

Properties (1)–(4) are easily verified. Property (5) follows from (1) and (2) (see [**2** or **6**]).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF RYUKYU, NISHIHARA, OKINAWA 903-01, JAPAN