SPECTRAL ASYMPTOTICS FOR SPINOR LAPLACIANS
AND MULTIPLICITIES
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ABSTRACT. We use Getzler's formula for the heat kernel of the spinor Laplacian to derive information about the asymptotic distribution of multiplicities in the quasi-regular representation of a semisimple Lie group $G$ modulo a cocompact discrete subgroup $\Gamma$.

0. Introduction. Let $G$ be a connected semisimple Lie group with finite center and let $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. It is well known that the right regular representation $R_\Gamma$ of $G$ on $L^2(\Gamma \backslash G)$ splits into a countable direct sum of irreducible unitary representations and that for each class $\pi \in \hat{\Gamma}$ occurring in this decomposition the multiplicity $N_\Gamma(\pi)$ is finite. The integers $N_\Gamma(\pi)$ have been, and still are, the subject of a great deal of investigation. One direction consists in studying their distribution as $\pi$ "approaches infinity" in $\hat{\Gamma}$. This is achieved by relating the $N_\Gamma(\pi)$ to the traces of suitable heat operators on $M = \Gamma \backslash G / K$ and then studying their asymptotic expansions. Here $K$ is a maximal compact subgroup of $G$. Also, we shall assume for simplicity that $\Gamma$ is torsion free and therefore $M$ is a smooth compact manifold.

Thus, using the connection Laplacian on $M$ naturally associated to a finite-dimensional representation $\tau$ of $K$, Wallach proves in [7] that
\begin{equation}
\sum_{\pi \in \hat{\Gamma}} N_\Gamma(\pi) \langle \pi_K, \tau \rangle e^{tx_\pi(\Omega)} = (4\pi t)^{-m/2} \dim(\tau) \text{vol}(M) + o(t^{-m/2}) \quad \text{as } t \to 0^+,
\end{equation}
where $\langle \pi_K, \tau \rangle$ is the intertwining number of $\pi$ restricted to $K$ and $\tau$, $x_\pi$ is the infinitesimal character of $\pi$, $\Omega$ stands for the Casimir element of $G$, $m = \dim(M)$ and $\text{vol}(M)$ is the volume of $M$ (with respect to a natural Riemannian structure).

In fact, one has an asymptotic expansion of the form
\begin{equation}
\sum_{\pi \in \hat{\Gamma}} N_\Gamma(\pi) \langle \pi_K, \tau \rangle e^{tx_\pi(\Omega)} \sim \sum_{k=0}^{\infty} a_k(\tau) t^{k-m/2} \quad \text{as } t \to 0^+;
\end{equation}
but, except when $\text{rank}(G/K) = 1$ (see [4]), not much is known about the coefficients $a_k(\tau)$, $k > 0$.

In this note we show that for $0 \leq k \leq m/2$ the coefficients $a_k$, viewed as functions of $\tau$, satisfy a series of equations which can be regarded as asymptotic analogues of the alternating sum formulas for multiplicities in [5]. This information is extracted from a limit formula of Getzler for the heat kernel of the Dirac operator (cf. [2, 3]).

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1. A consequence of Getzler’s formula. Given an inner product space $V$ over $\mathbb{R}$ we denote by $C(V)$ the associated Clifford algebra, defined as in [2], and by $\sigma: C(V) \to \wedge V$, $\theta: \wedge V \to C(V)$ the canonical isomorphisms (of vector spaces) between the Clifford algebra and the exterior algebra of $V$, with $\theta = \sigma^{-1}$. According to whether $m = \dim M$ is even or odd, $C(V) = C(V) \otimes_{\mathbb{R}} C$, resp $C^\text{ev}(V) = \theta(\wedge_{C}^{\text{ev}} V)$, is a simple algebra over $\mathbb{C}$ and thus has a unique simple module $S(V)$ of dimension $2^{m/2}$, resp. $2^{[(m+1)/2]}$, called the space of spinors over $V$. In what follows we shall identify $C(V)$, resp. $C^\text{ev}(V)$, with $\text{End } S(V)$. We recall that $S(V)$ can be equipped with an inner product such that each $v \in V$ acts as a selfadjoint operator on $S(V)$. We shall need the following elementary result, whose proof is similar to that of Theorem 1.8 in [2] and will be omitted.

**LEMMA.** Let $\omega \in \wedge_{C}^{2m} V$ and $h \in \text{End } S(V)$. Then

$\text{Tr}(\theta(\omega)h) = (-1)^{m} \dim S(V) \langle \sigma(h), \omega \rangle$.

We now consider a compact spin manifold $M$, of dimension $m$, and denote by $S(TM)$ its spin bundle. $\text{End } S(TM)$ will be identified with $C(TM)$ if $m$ is even, respectively with $C^\text{ev}(TM)$ when $m$ is odd. We shall also identify $TM$ and $T^*M$ via the Riemannian metric. The above isomorphisms $\sigma$ and $\theta$ will then induce bundle isomorphisms $\sigma: C(TM) \to \wedge_{C} T^*M$, $\theta: \wedge_{C} T^*M \to C(TM)$.

Let $E$ be a Hermitian vector bundle $M$, with connection $\nabla^E$. The Dirac operator $D_E$, acting on the space of $C^\infty$ sections $\Gamma(S(TM) \otimes E)$, is defined in terms of a local orthonormal frame $\{e_1, \ldots, e_m\}$ by the expression

$$D_E = \sum_{j=1}^{m} \theta(e_j) \otimes I(\nabla_{e_j}^S \otimes I + I \otimes \nabla_{e_j}^E),$$

where $\nabla^S$ is the connection on $S(TM)$ induced by the Riemannian connection on $TM$. With the present definition of the Clifford algebra, $D_E$ is skew-adjoint and therefore the associated heat semigroup is $e^{tD_E^2}$.

Let $K_t(x,y)$ be the Schwartz kernel of $e^{tD_E^2}$. For each $x \in M$, $K_t(x,x)$ is an endomorphism of $S(T_x M) \otimes E_x$ and so $\text{Tr}_{E_x} K_t(x,x) \in \text{End } S(T_x M)$. We now define a (nonhomogeneous) form $k_t(x) = \sum_{j=0}^{m} k^{(j)}(x)$ on $M$ by

$$k_t(x) = \sigma(\text{Tr}_{E_x} K_t(x,x)).$$

Getzler’s main result in [3] implies that

$$\lim_{\epsilon \to 0^+} \sum_{j=0}^{m} e^{m-j} k^{(j)}_{\epsilon}(x) = (4\pi t)^{-m/2} \text{ch}(tR_E)(x) \wedge \hat{A}(tR)(x),$$

where $R \in \Gamma(\wedge^2 T^*_CM \otimes \text{End } TM)$ is the Riemannian curvature of $M$, $R_E \in \Gamma(\wedge^2 T^*_CM \otimes \text{End } E)$ is the curvature of the connection $\nabla^E$, 

$$\hat{A}(R) = \det \left( \frac{R/2}{\sinh(R/2)} \right)^{1/2} \in \Gamma \left( \wedge^\text{ev} T^*_CM \right),$$

and

$$\text{ch}(R_E) = \text{Tr}(e^{R_E}) \in \Gamma \left( \wedge^\text{ev} T^*_CM \right).$$
Taking in both sides of (1.1) the scalar product with \( \overline{\omega}(x) \), where \( \omega \) is a 2\( l \)-form on \( M \), one obtains
\[
\lim_{t \to 0^+} t^{m-2l} (k_{\varepsilon x}(x), \overline{\omega}(x)) = (4\pi t)^{-m/2} t^{l} (\text{ch}(R_E)(x) \wedge \hat{A}(R)(x), \overline{\omega}(x)).
\]

Now, due to the above lemma, one can rewrite this as
\[
(1.2) \quad \lim_{t \to 0^+} t^{m/2-l} \text{Tr}((\theta(\omega)(x) \otimes I)K_t(x,x)) = (-1)^l C_m (\text{ch}(R_E)(x) \wedge \hat{A}(R)(x), \overline{\omega}(x)),
\]
where
\[
C_m = 2^{-[(m+1)/2]} \pi^{-m/2}.
\]

Since \( x \mapsto (\theta(\omega)(x) \otimes I)K_t(x,x) \in \text{End} S(T_x M) \otimes E_x \) is the diagonal restriction of the Schwartz kernel of the operator \( (\theta(\omega) \otimes I)e^{tD^2_E} \), by integrating (1.2) over \( M \) one obtains the following statement.

**Proposition.** For any 2\( l \)-form \( \omega \) on \( M \), one has
\[
(1.3) \quad \lim_{t \to 0^+} t^{m/2-l} \text{Tr}((\theta(\omega) \otimes I)e^{tD^2_E}) = (-1)^l C_m \int_M \text{ch}(R_E) \wedge \hat{A}(R) \wedge \star \omega.
\]

Let us note that for the 0-form \( \omega = 1 \) this is just Weyl’s formula for spinor Laplacians, while for \( l = m/2 \) (\( m \) even) and \( t^{-l} \omega = \) the volume form, one gets the index formula for Dirac operators.

2. Application to multiplicities. We shall now specialize the above result to the case of a compact locally symmetric space \( M = \Gamma\backslash G/K \), where \( G \), \( K \) and \( \Gamma \) are as in the introduction. Let \( g \) (resp. \( k \)) be the Lie algebra of \( G \) (resp. \( K \)) and let \( p \) be the orthogonal of \( k \) with respect to the Cartan-Killing form \( B \). We endow \( M \) with the Riemannian metric obtained from \( B|_p \times p \) via the identification of \( p \) with the tangent space at \( 0 = 1 \cdot K \) to \( G/K \).

Let \( \mathfrak{t} \) be a maximal abelian subalgebra of \( k \) and let \( \mathfrak{h} \) be the centrlizer of \( \mathfrak{t} \) in \( g \). Then \( \mathfrak{h} \) is a Cartan subalgebra of \( g \). Let \( \Phi \) be the root system of \( (g_C, h_C) \) and let \( \Phi_k \) be the root system of \( (k_C, t_C) \). We fix, once and for all, a set of positive roots \( \Psi_k \) for \( \Phi_k \) and a set of positive roots \( \Psi \) for \( \Phi \), in a compatible fashion. As usual, we denote by \( \rho, \rho_k \), and \( \rho_n \) the half-sum of the roots in \( \Psi, \Psi_k \), and \( \Psi_n = \Psi - \Psi_k \) respectively. Let \( T \) be the maximal torus of \( K \) with Lie algebra \( \mathfrak{t} \). The dual group \( \hat{T} \) will be identified, via exponentiation, to a lattice \( LT \subset \mathfrak{t}^* \).

The spin module associated to \( \langle p, \langle \cdot, \cdot \rangle \rangle \) will be denoted \( S \). It is, in particular, a \( \mathfrak{t} \)-module. We do not postulate the existence of a \( G \)-invariant spin structure on the symmetric space \( G/K \), and thus \( S \) need not be a \( K \)-module. Consider, however, an irreducible \( \mathfrak{t} \)-module \( V_{\nu} \) whose highest weight \( \nu \in \mathfrak{t}^* \) satisfies the condition
\[
(2.1) \quad \nu + \rho_n \in LT.
\]
Since every weight of \( S \) differs from \( \rho_n \) by a sum of roots, (2.1) is easily seen to guarantee the fact that the representation of \( \mathfrak{t} \) on \( S \otimes V_{\nu} \) lifts to a representation of \( K \). This representation of \( K \), in turn, gives rise first to a homogeneous bundle over \( G/K \) and then, by passing to \( \Gamma \)-orbits, to a vector bundle over \( M \) which will be denoted \( S(M, \nu) \). This bundle comes equipped with a Hermitian structure, induced by the \( K \)-invariant inner product on \( S \otimes V_{\nu} \), and with a unitary connection \( \nabla^{(S, \nu)} \), inherited from that of the principal bundle \( G \to G/K \) (defined by the splitting.
\[ g = \mathfrak{k} \oplus \mathfrak{p} \). We can therefore form the twisted Dirac operator \( D_\nu : \Gamma(S(M, \nu)) \to \Gamma(S(M, \nu)) \). Explicitly, after identifying \( \Gamma(S(M, \nu)) \) with the subspace \( (C^\infty(\Gamma/G) \otimes S \otimes V_\nu)^K \) of all \( K \)-invariant elements in \( C^\infty(\Gamma/G) \otimes S \otimes V_\nu \), \( D_\nu \) is given by the formula

\[ D_\nu = \sum_{j=1}^m R_\Gamma(X_j) \otimes \theta(X_j) \otimes I, \]

where \( \{X_1, \ldots, X_m\} \) is an orthonormal basis for \( \mathfrak{p} \). Moreover (see [6], \S 3),

\[ -D_\nu = -R_\Gamma(\Omega) \otimes I \otimes I + (\|\nu + \rho_k\|^2 - \|\rho\|^2)I \otimes I \otimes I, \]

where \( \Omega \) is the Casimir element of \( g \).

We recall that \( \wedge p_C^* \) (resp. \( \wedge^{ev} p_C^* \), if \( m \) is odd) and \( \text{End}(S) \) are isomorphic as \( \text{SO}(\mathfrak{p}) \)-modules. In particular \( \omega \in \wedge^{ev} p_C^* \) is \( K \)-invariant if and only if \( \theta(\omega) \in \text{End}_K(S) \).

Consider now a unitary representation \( \pi \) of \( G \) on a Hilbert space \( \mathcal{H}_\pi \). Then

\[ I \otimes \theta(\omega) \otimes I : \mathcal{H}_\pi \otimes S \otimes V_\nu \to \mathcal{H}_\pi \otimes S \otimes V_\nu \]

commutes with the (tensor product) action of \( K \), and therefore restricts to an operator

\[ \theta_{\pi,\nu}(\omega) : (\mathcal{H}_\pi \otimes S \otimes V_\nu)^K \to (\mathcal{H}_\pi \otimes S \otimes V_\nu)^K, \]

where the superscript \( K \) signifies passage to \( K \)-invariant elements. Note that if \( \pi \) is irreducible, \( (\mathcal{H}_\pi \otimes S \otimes V_\nu)^K \) has finite dimension.

Let \( r \in \wedge^2 p_C^* \otimes \text{End}(\mathfrak{p}) \) and \( r_\nu \in \wedge^2 p_C^* \otimes \text{End}(V_\nu) \) be defined as follows

\[ r = - \sum_{1 \leq i, j \leq m} \text{ad}[X_i, X_j] \otimes \xi_i \wedge \xi_j, \]

\[ r_\nu = - \sum_{1 \leq i, j \leq m} r_\nu[X_i, X_j] \otimes \xi_i \wedge \xi_j, \]

where \( \{X_1, \ldots, X_m\} \) is an orthonormal basis of \( \mathfrak{p} \), \( \{\xi_1, \ldots, \xi_m\} \) is its dual basis for \( \mathfrak{p}^* \), and \( r_\nu \) denotes the representation of \( K \) on \( V_\nu \). It is easy to check that \( r \) and \( r_\nu \) are independent of the orthonormal basis, and also that they are \( K \)-invariant. We then form

\[ \hat{A}(r) = \text{det} \left( \frac{r/2}{\sinh(r/2)} \right)^{1/2} \in \left( \bigwedge^{ev} p_C^* \right)^K \]

and

\[ \text{ch}(r_\nu) = \text{Tr}_{V_\nu}(e^{r_\nu}) \in \left( \bigwedge^{ev} p_C^* \right)^K. \]

Finally, we denote

\[ \tilde{G}_{\Gamma, \nu} = \{ \pi \in \hat{G} ; N_\Gamma(\pi) > 0, \dim(\mathcal{H}_\pi \otimes S \otimes V_\nu)^K > 0 \} \]

and recall that \( \chi_\pi \) stands for the infinitesimal character of \( \pi \in \hat{G} \).

**Theorem.** For each \( \omega \in (\wedge^2 p_C^*)^K \), one has

\[ \lim_{t \to 0^+} t^{m/2 - l} \sum_{\pi \in \tilde{G}_{\Gamma, \nu}} N_\Gamma(\pi) \text{Tr} \theta_{\pi,\nu}(\omega)e^{\chi_\pi(\Omega)t} = (-1)^l C_m(\text{ch}(r_\nu) \wedge \hat{A}(r), \omega) \text{vol}(\Gamma \backslash G/K). \]
PROOF. Let us assume for the moment that the $\epsilon$-action on $S$ does lift to $K$ and therefore $M$ is a spin manifold. $S(TM)$ is then the bundle induced by $S$. We denote by $E_{\nu}$ the bundle on $M$ induced by $V_{\nu}$ and by $\tilde{\omega}$ the form on $M$ whose lift to $G/K$ is the $G$-invariant form determined by $\omega$. Clearly, $\theta(\tilde{\omega}) \in \text{End} S(TM)$ is just the endomorphism induced by $\theta(\omega) \in \text{End}_K(S)$. Furthermore, under the identification of $\Gamma(S(TM) \otimes E_{\nu})$ with $(C^\infty(\Gamma\backslash G) \otimes S \otimes V_{\nu})^K$, the multiplication operator $\theta(\omega) \otimes I$ becomes

$$\theta_{R_\tau, \nu}(\omega) = I \otimes \theta(\omega) \otimes I(\varpi^2(\Gamma\backslash G) \otimes S \otimes V_{\nu})^K.$$  

Let us also note that, since the Riemannian connection on $TM$ coincides with that induced by the canonical $G$-invariant connection on $T(G/K)$, the Dirac operator $D_{R_\tau}$ is precisely the operator $D_{\nu}$ given by $(2.2)$.

From the decomposition of $R_\tau$ into irreducible components,

$$L^2(\Gamma\backslash G) = \sum_{\pi \in \tilde{\pi}} N_{\Gamma}(\pi)\pi_\pi,$$

it follows that

$$(L^2(\Gamma\backslash G) \otimes S \otimes V_{\nu})^K \cong \sum_{\pi \in \tilde{\pi}} N_{\Gamma}(\pi)(\pi_\pi \otimes S \otimes V_{\nu})^K.$$  

So, each $\pi \in \tilde{\pi}$ contributes $N_{\Gamma}(\pi)$ summands of the form $(\pi_\pi \otimes S \otimes V_{\nu})^K$; in turn, each of these summands is invariant under both $D_{\nu}^2$ and $\theta_{R_\tau, \nu}(\omega)$. Moreover, one has:

$$D_{\nu}^2(\pi_\pi \otimes S \otimes V_{\nu})^K = (\chi_\tau(\Omega \times - (\|\nu + \rho_k\|^2 - \|\rho\|^2)) I \quad (\text{cf. (2.3)}).$$

and

$$\theta_{R_\tau, \nu}(\omega)(\pi_\pi \otimes S \otimes V_{\nu})^K = \theta_{\pi, \nu}(\omega).$$

Therefore,

$$\text{Tr}(\theta_{R_\tau, \nu}(\omega)e^{tD^2}) = e^{(\|\rho\|^2 - \|\nu + \rho_k\|^2)t} \sum_{\pi \in \tilde{\pi}} N_{\Gamma}(\pi) \text{Tr} \theta_{\pi, \nu}(\omega)e^{\chi_\tau(\Omega)t},$$

and thus $(2.4)$ follows immediately from $(1.3)$. Finally, let us drop the assumption that $S$ integrates to a representation of $K$. The role of the bundle $S(TM) \otimes E$ is then assumed by $S(M, \nu)$, the bundle induced by the $K$-module $S \otimes V_{\nu}$. Now $\text{End}(\varpi^2(\Gamma\backslash G) \otimes S \otimes V_{\nu}) = \text{End}(S) \otimes \text{End}(V_{\nu})$ and, unlike $S$ and $V_{\nu}$ which are only $\mathfrak{g}$-modules, both $\text{End}(S)$ and $\text{End}(V_{\nu})$ are in fact $K$-modules. With this observation it is easy to see that the arguments in §1 still apply, giving $(1.3)$ and therefore $(2.4)$. \qed

When $\omega = 1$, $(2.4)$ gives no new information beyond that coming from $(0.1)$. At the opposite extreme, when $l = m/2$ (assuming $m$ is even) and $i^{-l} \omega$ is the invariant volume form $v$ on $G/K$, the left-hand side of $(2.4)$ is independent of $t$ and coincides with the index of $D_{\nu}$. Thus (see also [1, §1]), $(2.4)$ becomes

$$\sum_{\pi \in \tilde{\pi}} N_{\Gamma}(\pi)(\dim(\pi_\pi \otimes S^+ \otimes V_{\nu})^K - \dim(\pi_\pi \otimes S^- \otimes V_{\nu})^K) = C_m i^{m/2}((\chi_\tau(\nu) \wedge \hat{A}(\nu))(m), v) \text{vol}(\Gamma\backslash G/K),$$

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which is essentially Miatello's alternating sum formula for multiplicities, associated to the homogeneous symbol of $D_v$ (cf. [4]). In general, (2.4) can be viewed as interpolating between the two extremes.

Let us finally interpret the theorem in terms of the coefficient functions of the asymptotic expansion (0.2). Let

$$S = \sum_{i=1}^{N} S_i, \quad \sigma = \sum_{i=1}^{N} \sigma_i$$

be the decomposition of $S$, as a $\mathfrak{k}$-module, into irreducible components, and let $P_i$ denote the orthogonal projection in $\text{End}_K(S)$ associated to $S_i, 1 \leq i \leq N$. Then

$$\theta(\omega) = \sum_{i=1}^{N} c_i(\omega) P_i$$

with

$$c_i(\omega) = \text{Tr}(\theta(\omega) P_i) / \dim S_i.$$

Hence

$$\theta_{\pi,\nu}(\omega) = \sum_{i=1}^{N} c_i(\omega) I \otimes P_i \otimes I |(n_{\pi} \otimes S \otimes V_\nu)^K$$

and therefore

$$\text{Tr} \theta_{\pi,\nu}(\omega) = \sum_{i=1}^{N} c_i(\omega) \langle \pi_K^*, \sigma_i \otimes \tau_\nu \rangle.$$

In fact each irreducible constituent $\sigma_i$ of the spin representation $\sigma$ is known to be of the form $\tau_{sp-Pk}$, where $s \in W^1 = \{\omega \in W(\mathfrak{g}_C, \mathfrak{h}_C); \omega \Psi \supset \Psi_k\}$. We can thus rewrite the last equality as

$$\text{Tr} \theta_{\pi,\nu}(\omega) = \sum_{s \in W^1} c_s(\omega) \langle \pi_K^*, \tau_{sp-Pk} \otimes \tau_\nu \rangle,$$

where now

$$c_s(\omega) = \sum_{i \in I_s} c_i(\omega), \quad \text{with } I_s = \{i; \sigma_i = \tau_{sp-Pk}\}.$$ 

Using (2.5) and also the fact that $N_\Gamma(\pi^*) = N_\Gamma(\pi), \chi_{\pi^*}(\Omega) = \chi_\pi(\Omega)$, the left-hand side of (2.4) becomes

$$\lim_{t \to 0^+} t^{m/2-l} \sum_{s \in W^1} c_s(\omega) \sum_{\pi \in \hat{G}_{\Gamma,\nu}} N_\Gamma(\pi) \langle \pi_K^*, \tau_{sp-Pk} \otimes \tau_\nu \rangle e^{\chi_\pi(\Omega) t}.$$ 

Thus, the information given by the theorem for the coefficients of the negative powers of $t$ in the asymptotic expansion (0.2) amounts to the following family of equations.

**COROLLARY.** For each $\omega \in (\Lambda^{2l} p_C^*)^K$, one has

$$\sum_{s \in W^1} c_s(\omega) a_j(\tau_{sp-Pk} \otimes \tau_\nu) = 0, \quad \text{if } 0 \leq j \leq l - 1,$$

and

$$\sum_{s \in W^1} c_s(\omega) a_l(\tau_{sp-Pk} \otimes \tau_\nu) = (-1)^l C_m (\text{ch}(\tau_\nu) \wedge \hat{\Delta}(r), \overline{\omega}) \text{vol}(\Gamma \backslash G/K).$$
REFERENCES


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