EMBEDDING COUNTABLE RINGS
IN 2-GENERATOR RINGS
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ABSTRACT. A short elementary proof involving matrices is used to show that any countable ring can be embedded in a 2-generator ring. Immediate corollaries are the known results that any countable (respectively finite) semigroup can be embedded in a 2-generator (respectively finite 2-generator) semigroup.

The object of this note is to show that any countable ring can be embedded in a 2-generator ring, using a short elementary proof involving matrices. An immediate corollary of our approach is the known result that any countable semigroup can be embedded in a 2-generator semigroup. The latter was first established by Evans [1] in 1952 using free semigroups, and since then a number of other proofs have been given (e.g. [3, 4, 5]). Interest in these embedding theorems was sparked by the famous Higman, Neumann, and Neumann paper [2] in 1949, which used free products to show that any countable group can be embedded in a 2-generator group. That this type of result does not hold in all algebraic systems is shown by the fact that the only countable abelian groups which can be embedded in 2-generator abelian groups are those which are the direct sum of two cyclic groups.

For a ring \( R \) we let \( R^\infty \) be the ring of all countably-infinite, column-finite matrices over \( R \) (that is, \( \mathbb{N}_0 \times \mathbb{N}_0 \) matrices with only a finite number of nonzero entries in each column, and with the usual matrix addition and multiplication). A ring need not have an identity. Note, however, that if \( R \) has an identity so also does \( R^\infty \).

**THEOREM.** Given any countable ring \( R \), there exists a ring \( T \) and a ring embedding \( \theta : R \to T \) such that \( \theta(R) \) is contained in a 2-generator subsemigroup of the multiplicative semigroup of \( T \).

**PROOF.** Step 1. Embedding \( R \) in a 3-generator subsemigroup. We can assume \( R \) has an identity because the standard construction for adding one preserves countability. Suppose \( R = \{a_1, a_2, \ldots, a_n, \ldots\} \). Let \( S = R^\infty \) and consider the ring embedding

\[
\psi : R \to S, \quad \psi(r) = \begin{pmatrix} r & 0 \\ 0 & \ddots \end{pmatrix}
\]
Let
\[
\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ & 0 & \cdots & \cdots \\ \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \\ \end{pmatrix},
\]
each a member of \( R_{\infty} \). Observe that right multiplication of a matrix by \( \beta \) shifts the columns one step to the left, while right multiplication by \( \gamma \) leaves the first column unchanged and sets all other columns to 0. Hence
\[
\psi(a_n) = \begin{pmatrix} a_n \\ 0 \\ 0 \\ \end{pmatrix} = \alpha \beta^{n-1} \gamma
\]
for all \( n \), provided we interpret \( \alpha \beta^0 \gamma \) as \( \alpha \gamma \). This shows \( \psi(R) \subseteq \langle \alpha, \beta, \gamma \rangle \), where \( \langle \alpha, \beta, \gamma \rangle \) is the multiplicative subsemigroup of \( S \) generated by \( \alpha, \beta, \gamma \).

**Step 2. Going from 3 (or \( n < \infty \)) generators to 2.** Let \( T = S_4 \) be the ring of \( 4 \times 4 \) matrices over \( S \), and again consider the “corner” embedding
\[
\phi: S \to T, \quad \phi(s) = \begin{pmatrix} s & 0 \\ 0 & 0 \\ 0 & 0 \\ \end{pmatrix} \in S_4.
\]
Let
\[
\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ \end{pmatrix}, \quad \xi = \begin{pmatrix} \alpha & \beta & \gamma & 1 \\ \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \end{pmatrix} \in T.
\]
By noting the effect of right multiplication by \( \eta \) and \( \delta \) (similar to \( \beta \) and \( \gamma \)) we have
\[
\xi \eta^3 = \delta \in \langle \eta, \xi \rangle
\]
and
\[
\phi(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} = \xi\delta, \quad \phi(\beta) = \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} = \xi\eta\delta;
\]
\[
\phi(\gamma) = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} = \xi\eta^2\delta.
\]
Thus \(\phi(\psi(R)) \subseteq \langle \eta, \xi \rangle\), whence \(\theta = \phi \circ \psi : R \to T\) is the desired ring embedding. The proof is complete. □

Presented with any countable semigroup \(S\), we can always embed \(S\) in the multiplicative semigroup of some countable ring \(R\) (for example, the semigroup ring \(\mathbb{Z}[S]\)). Hence we have

**COROLLARY.** Every countable ring can be embedded in a 2-generator ring. Every countable semigroup can be embedded in a 2-generator semigroup. □

There are two further bonuses of our proof, for it shows:

1. Every countable-dimensional algebra over a field can be embedded in a 2-generator algebra (in Step 1 take \(\{a_1, a_2, \ldots, a_n, \ldots\}\) to be a basis for \(R\)).

2. Every finite ring (respectively finite semigroup) can be embedded in a 2-generator finite ring (respectively 2-generator finite semigroup). This follows from Step 2 alone, for a general \(n\), using the obvious \((n + 1) \times (n + 1)\) matrices. The semigroup version of this was obtained by Neumann [3].

**ADDED IN PROOF.** The author has recently learned that the above Corollary for rings and the result (1) were obtained by A. I. Mal’tsev in *A representation of nonassociative rings*, Uspehi. Mat. Nauk 7 (1955), 181–185. Mal’tsev’s proof uses free rings and is similar to Evans’ proof [1] for semigroups. Also using free rings, V. Ya. Belyaev has shown in *Subrings of finitely presented associative rings*, Algebra i Logika 17 (1978), 627–638, that any (associative) ring with a recursively enumerable set of defining relations can be embedded in a 2-generator finitely presented ring.

The techniques of the present paper can be modified to show that any countable (respectively finite) ring with identity can be embedded in a 2-generator (respectively finite 2-generator) ring with identity such that the embedding preserves the identity and respects the centers. (Unlike the Theorem, however, the embedding is no longer into a 2-generator multiplicative subsemigroup.) This will appear in a paper *Identity-preserving embeddings of countable rings into 2-generator rings* by the author, C. E. Vinsonhaler, and W. J. Wickless. The embeddings used by Mal’tsev and Belyaev (above) do not preserve the identity.

**REFERENCES**


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