AN INEQUALITY FOR EIGENVALUES OF STURM-LIOUVILLE PROBLEMS

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ABSTRACT. By means of simple transformations, inequalities for eigenvalues corresponding to different boundary conditions are derived.

1. Let \( p(x) > 0 \) be a continuous function and consider the following eigenvalue problems:

\[
(1.1) \quad \phi''(x) + \lambda p(x) \phi(x) = 0 \quad \text{in } (-1,1), \quad \phi(-1) = \phi(1) = 0,
\]
and

\[
(1.2) \quad \psi''(x) + \mu p(x) \psi(x) = 0 \quad \text{in } (-1,1), \quad \psi'(-1) = \psi'(1) = 0.
\]

It is well known that there exist two countable sequences of eigenvalues \( 0 < \lambda_1 < \lambda_2 < \cdots \) and \( 0 = \mu_1 < \mu_2 < \cdots \) both tending to infinity as \( n \to \infty \). According to Poincaré's principle they may be characterized as

\[
(1.3) \quad \lambda_n = \inf_{L^n_1} \frac{\max_{L^n_1} R[v]}{\max_{L^n_1} v} \quad \text{and} \quad \mu_n = \inf_{L^n_1} \frac{\max_{L^n_1} R[v]}{\max_{L^n_1} v},
\]

where

\[
R[v] := \int_{-1}^{1} v'^2 dx / \int_{-1}^{1} v^2 p dx
\]
is the Rayleigh quotient, \( L^n_1 \subset H^1_0(-1,1) \) and \( L^n_1 \subset H^1(-1,1) \) are \( n \)-dimensional function spaces. In (1.3), equality holds if and only if \( v \) is the \( n \)th eigenfunction.

2. From (1.3) it follows immediately that

\[
(2.1) \quad \mu_n < \lambda_n.
\]

The aim of this note is to sharpen inequality (2.1) for a special class of mass densities \( \rho \).

3. Let us start with two elementary lemmas. The first observation is essentially due to [3].

LEMMA 1. Let \( \phi_1 \) and \( \phi_n \) be the first and the \( n \)th eigenfunctions of (1.1). Then \( v_n(x) = \phi_n(x) / \phi_1(x) \) is the \( n \)th eigenfunction of

\[
(3.1) \quad \left\{ \phi_1^2 v_n' \right\}' + \bar{\omega} \phi_1^2 \rho v_n = 0 \quad \text{in } (-1,1), \quad v_n'(-1) = v_n'(1) = 0,
\]
with the corresponding eigenvalue \( \bar{\omega}_n = \lambda_n - \lambda_1 \).

PROOF. Since \( \phi_k \) has the expansion \( \phi_k(x) = \phi_k'(-1)(x + 1) + o((x + 1)^2) \), \( \phi_k'(-1) \neq 0 \), it follows that

\[
v'(x) = \frac{\phi_n'(x) - \phi_n(x) \phi_1'(x)}{\phi_1^2(x)} = O(x + 1),
\]
which implies \( v'(-1) = 0 \). Similarly we have \( v'(1) = 0 \). The remaining part is verified by a straightforward computation.

The next result is a slightly modified version of a lemma by Payne and Weinberger [2].

**Lemma 2.** Let \( v_n \) be the \( n \)th eigenfunction of (3.1). Then \( w_{n-1} := v'_n \phi_1 \) is the \((n - 1)\)st eigenfunction of

\[
\left( \frac{w'}{\rho} \right)' - \frac{1}{\rho} \left[ \frac{2\phi_1'^2}{\phi_1^2} + \frac{\phi_1 \rho'}{\phi_1 \rho} \right] w + \nu w = 0 \quad \text{in} \ (-1, 1),
\]

\[
w(-1) = w(1) = 0,
\]

with the corresponding eigenvalue \( \nu_{n-1} = \lambda_n - 2\lambda_1 \).

The proof of (3.2) follows from a straightforward computation. In view of Poincaré’s variational principle we have

\[
\nu_{n-1} = \inf_{L'_{n-1}} \max_{v \in L_{n-1}} \left\{ \int_{-1}^{1} \frac{v'^2}{\rho} \, dx + \int_{-1}^{1} \left[ \frac{2\phi_1'^2}{\phi_1^2} + \frac{\phi_1 \rho'}{\phi_1 \rho} \right] \frac{v^2}{\rho} \, dx \right\} / \int_{-1}^{1} v^2 \, dx,
\]

where \( L'_{n-1} \) is an \((n - 1)\)-dimensional space of functions in \( H^1_0(-1, 1) \) for which (3.3) makes sense. If the assumption

\[
2\frac{\phi_1'^2}{\phi_1^2} + \frac{\phi_1 \rho'}{\phi_1 \rho} \geq 0
\]

holds, then we have by (3.3)

\[
\nu_{n-1} \geq \omega_{n-1},
\]

where \( \omega_{n-1} \) is the \((n - 1)\)st eigenvalue of

\[
\left( \frac{u'}{\rho} \right)' + \omega u = 0 \quad \text{in} \ (-1, 1), \quad u(-1) = u(1) = 0.
\]

**Lemma 3.** \( \omega_{n-1} = \mu_n \), where \( \mu_n \) is the \( n \)th eigenvalue of (1.2).

**Proof.** Put \( u' / \rho = \psi \). Then \( \psi' = -\omega u \) and \( \psi'' + \omega \rho \psi = 0 \), which proves the assertion.

We are now ready to establish our main result.

**Theorem 1.** Under the assumptions

(i) \( \rho(-x) = \rho(x) \), and

(ii) \( \rho(x) \) increasing in \((-1, 0)\),

we have \( \lambda_n - 2\lambda_1 \geq \mu_n \).

**Proof.** We first consider the case where \( \rho \) is differentiable. In view of assumption (i), \( \phi_1 \) is symmetric with respect to \( x = 0 \). Moreover \( |\phi_1(x)| \) is concave, hence \( \phi_1' \rho / \phi_1 \geq 0 \) in \((-1, 1)\). Thus assumption (A) is satisfied. The assertion now follows from (3.4) together with Lemmas 2 and 3. If \( \rho \) is not differentiable we approximate \( \rho \) by means of differentiable functions \( \{\rho_n\}_{n=1}^{\infty} \) and use the fact that \( \lambda_n \) and \( \mu_n \) depend continuously on \( \rho \) [1, p. 418].
4. Consider the eigenvalue problem with mixed boundary conditions

\[(4.1) \quad \phi''(x) + \Lambda \rho(x) \phi(x) = 0 \quad \text{in} \ (0,1), \quad \phi(0) = \phi'(1) = 0.\]

It is easily seen that for the first eigenfunction \(\phi_1\) we have \(\phi_1 \phi'_1 > 0\) in \((0,1)\). If

(iii) \(\rho(x)\) is increasing in \((0,1)\),

then condition (A) is satisfied. The same arguments as for Theorem 1 yield

**Theorem 2.** Assume (iii). Then \(\Lambda_n - 2\Lambda_1 \geq \mu_n\).

We finally consider the more general eigenvalue problem

\[(4.2) \quad (\sigma(x)\phi'(x))' + \lambda \rho(x) \phi(x) = 0 \quad \text{in} \ (-1,1)\]

with either boundary conditions

\[(4.3) \quad \phi(-1) = \phi(1) = 0,\]

\[(4.4) \quad \phi(-1) = \phi'(1) = 0,\]

or

\[(4.5) \quad \phi'(-1) = \phi'(1) = 0.\]

If we introduce the new variable \(t = \int_1^x (ds/\sigma(s))\), (4.2) becomes

\[(4.6) \quad \ddot{\phi} + \lambda \sigma(x(t)) \rho(x(t)) \phi = 0.\]

A direct application of Theorems 1 and 2 yields obviously

**Theorem 3.** Let \(\lambda_n, \mu_n, \Lambda_n\) be the eigenvalues of (4.2) with the boundary equations (4.3), (4.4), or (4.5) respectively. Under the assumptions

(i)' \(\sigma \rho(-x) = \sigma \rho(x)\),

(ii)' \(\sigma \rho\) increasing in \((-1,0)\),

we then have \(\lambda_n - 2\lambda_1 \geq \mu_n\).

Under the assumption

(iii)' \(\sigma \rho\) increasing in \((-1,1)\),

we then have \(\Lambda_n - 2\Lambda_1 \geq \mu_n\).