

CONVOLUTION EQUATIONS IN SPACES OF DISTRIBUTIONS SUPPORTED BY CONES

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ABSTRACT. We describe some examples of surjective convolutors on $\mathcal{D}'(\Gamma)$, for Γ a closed convex cone in \mathbf{R}^n . We also give necessary and sufficient conditions on S_1, \dots, S_m in $\mathcal{S}'(\Gamma)$ to be generators of the whole convolution algebra $\mathcal{S}'(\Gamma)$.

0. Introduction. Let $\Gamma \subset \mathbf{R}^n$ be a closed convex cone, whose vertex is the origin, and which does not contain any straight line. Denote by $\mathcal{D}'(\Gamma)$ the space of distributions supported in Γ ; it is well known that $\mathcal{D}'(\Gamma)$ is a convolution algebra, so if $S \in \mathcal{E}'(\mathbf{R}^n)$ is a distribution with compact support contained in Γ , then $S * \mathcal{D}'(\Gamma) \subseteq \mathcal{D}'(\Gamma)$ (see, e.g., [6, p. 33]). By now, a complete characterization (in terms of the decrease of \hat{S} , the Fourier transform of S) of those S for which $S * \mathcal{D}'(\Gamma) = \mathcal{D}'(\Gamma)$ is well known (see, e.g., [4, pages 356 and following]). On the other hand, in a recent paper, [8] a characterization of a completely different nature is given, at least for $n = 1$ (and hence $\Gamma = [0, +\infty)$); this characterization, unfortunately, is of difficult application, as it requires the test of the singular support of $S * u$ for an extremely large class of distributions u and, as a consequence, no nontrivial examples are given in [8].

The purpose of this paper is twofold: first, in §1, we show that the condition of Shambayati and Zielezny [8] can be replaced by an equivalent one, in which only compactly supported distributions are used, and which has the advantage of being quite easy to check; in particular we are able to use our condition to provide several examples of distributions S in $\mathcal{E}'(\mathbf{R})$ for which $S * \mathcal{D}'([0, +\infty)) = \mathcal{D}'([0, +\infty))$. One can now use this characterization, together with the well-known results of Hörmander on hyperbolic convolution equations, to deduce nontrivial properties of the Fourier transform of those invertible distributions $S \in \mathcal{E}'(\mathbf{R})$ for which $\text{sing supp}(S) = \{0\}$, and of other classes of examples. In this section we also prove that the original characterization provided in [8] can indeed be extended to the case $n > 1$; here, however, as we show with a simple example taken from the theory of constant coefficients partial differential operators, the condition cannot be eased.

In §2, finally, we deal with the following related problem: let $\mathcal{S}'(\Gamma)$ denote the space of tempered distributions, with support contained in the cone $\Gamma \subset \mathbf{R}^n$, and let S_1, \dots, S_m be m distributions in $\mathcal{S}'(\Gamma)$; we provide a necessary and sufficient condition on (S_1, \dots, S_m) in order to ensure the existence of m distributions $E_1, \dots, E_m \in \mathcal{S}'(\Gamma)$ such that $S_1 * E_1 + \dots + S_m * E_m = \delta$; for $m = 1$ the condition was well known (see [7], but also [9 and 10]), while for $m > 1$ we prove it with techniques similar to those used by Hörmander in [3]: our method, indeed, uses L^2 -estimates for solutions of the Cauchy-Riemann equation, together with the

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characterization of the space $\hat{S}'(\Gamma)$, provided in [9], and could be suitably modified to yield the solution for a similar problem in $\mathcal{D}'_F(\Gamma)$, the space of finite-order distributions with support in Γ .

1. According to Ehrenpreis [2], we say that a compactly supported distribution $S \in \mathcal{E}'(\mathbf{R}^n)$ is invertible if, with \hat{S} denoting the Fourier transform of S , there are constants $A_1, A_2, A_3 > 0$ such that for every $\xi \in \mathbf{R}^n$ there exists $\eta \in \mathbf{R}^n$ satisfying

$$|\xi - \eta| \leq A_1 \log(2 + |\xi|) \quad \text{and} \quad |\hat{S}(\eta)| \geq (A_2 + |\xi|)^{-A_3}.$$

It is well known that a distribution S in $\mathcal{E}'(\mathbf{R}^n)$ is invertible if and only if $S * \mathcal{D}'(\mathbf{R}^n) = \mathcal{D}'(\mathbf{R}^n)$.

If K is a compact convex set in \mathbf{R}^n , $H_K(\xi) = \sup_{x \in K} \langle x, \xi \rangle$ will denote its supporting function; denote by \mathcal{K} the set of all supporting functions of compact convex sets (including the empty set, whose supporting function is $-\infty$): then, if $S \in \mathcal{E}'(\mathbf{R}^n)$, $\mathcal{K}(S)$ denotes the set of all $h \in \mathcal{K}$ such that there is a sequence $\xi_\nu \rightarrow \infty$ in \mathbf{R}^n with

$$L_S(z, \xi_\nu) = \frac{\log |\hat{S}(\xi_\nu + z \log |\xi_\nu|)|}{\log |\xi_\nu|}$$

converging to a plurisubharmonic function with supporting function h [4].

Let us now restrict our attention to the case $n = 1$; let $S \in \mathcal{E}'(\mathbf{R})$ be a distribution with compact support, $\text{supp}(S) \subseteq [0, +\infty)$; let $\mathcal{D}'([0, +\infty))$ be the space of distributions with supports contained in $[0, +\infty)$ and \mathcal{D}'_+ the space of distributions with supports contained in $[a, +\infty)$, for some a in \mathbf{R} . We have:

THEOREM 1. *Let $S \in \mathcal{E}'(\mathbf{R}), \text{supp}(S) \subseteq [0, +\infty)$. Then $S * \mathcal{D}'([0, +\infty)) = \mathcal{D}'([0, +\infty))$ if and only if S is invertible and*

(1) *for all j in \mathbf{Z} , for all a in \mathbf{R} , there exists b in \mathbf{R} such that for all $h \in \mathcal{K}(S)$ and for all $x \in \mathbf{R}$*

$$h(\xi) + x \cdot \xi \leq H_{[j, a]}(\xi), \text{ for all } \xi, \text{ implies } x \in [j, b].$$

PROOF. In view of Corollary 1 of [8], we only have to show that (1) is equivalent to

(2) for all $j \in \mathbf{Z}$ and all $u \in \mathcal{D}'_+$, $\text{sing supp}(S * u) \subseteq [j, +\infty)$ implies $\text{sing supp}(u) \subseteq [j, +\infty)$.

We first notice that condition (2) is equivalent to

(3) for all $j \in \mathbf{Z}$ and all $u \in \mathcal{E}'$, $\text{sing supp}(S * u) \subseteq [j, +\infty)$ implies $\text{sing supp}(u) \subseteq [j, +\infty)$.

It is clear that (2) implies (3), as $\mathcal{E}' \subset \mathcal{D}'_+$. On the other hand, let (3) hold, $u \in \mathcal{D}'_+$ and $\text{sing supp}(S * u) \subset [j, +\infty)$ for some j . Let $\alpha = \inf\{x: x \in \text{supp}(u)\}$: if $\alpha \geq j$ we do not need to prove anything, otherwise take $\phi \in C_0^\infty(\mathbf{R})$, $\phi \equiv 1$ on $[\alpha, j]$, $\phi \geq 0$, and define

$$T_1 = \phi \cdot u \text{ and } T_2 = (1 - \phi) \cdot u,$$

so that $u = T_1 + T_2$. Notice that $\text{supp}(T_2) \subseteq [j, +\infty)$, hence $\text{sing supp}(T_2) \subseteq [j, +\infty)$. Now

$$S * T_1 = S * u - S * T_2;$$

notice that both $S * u$ and $S * T_2$ are C^∞ in $(-\infty, j)$, because $\text{supp}(T_2) \subseteq [j, +\infty)$ and $\text{supp}(S) \subseteq [0, +\infty)$; therefore

$$\text{sing supp}(S * T_1) \subseteq [j, +\infty).$$

Now T_1 is of compact support, and hence (3) implies that $\text{sing supp}(T_1) \subseteq [j, +\infty)$. Finally, as $u = T_1 + T_2$, we get that $\text{sing supp}(u) \subseteq [j, +\infty)$. The equivalence of (3) and (1) is now an immediate consequence of Theorem 16.3.13 of [4].

Condition (1) or, equivalently, condition (3), enables us to give some classes of examples of surjective convolutors onto $\mathcal{D}'([0, +\infty))$; none of them were given in [8].

EXAMPLE 1. Let S be a compactly supported invertible distribution with $\text{sing supp}(S) = \{0\}$; then we can either use Corollary 16.3.15 of [4], or remark that, in this case, the set $\mathcal{H}(S)$ contains only one function, namely the supporting function of the convex hull of the singular support of S ; in both ways one sees that (1) is satisfied, and therefore $S*$ is a surjection.

EXAMPLE 2. Suppose \hat{S} is an exponential polynomial (this means that S is a distribution with finite support) such that $0 \in \text{sing supp}(S)$. Then it is well known that S is invertible and Corollary 16.3.18 of [4] implies that (1) is satisfied.

EXAMPLE 3. Let $\mu, \nu \in \mathcal{E}'(\mathbf{R})$; suppose that μ contains an atom, $\text{sing supp}(\mu) \cap \text{sing supp}(\nu) = \emptyset$, and that $\mathcal{H}(\mu)$ and $\mathcal{H}(\nu)$ only contain one function. Then if $S = \nu + P(D)\mu$, it is shown in [1] that S is invertible and in [4] that $\mathcal{H}(S)$ contains only one function, and therefore S induces, by convolution, a surjection onto $\mathcal{D}'([0, +\infty))$. In particular, one could take ν to have a singular support constituted only by isolated points; see Corollary 16.3.18 in [4].

REMARK. In case $\mathcal{H}(S)$ contains only one function, (1) implies that

$$\text{sing supp}(S * u) \subseteq [j, a] \text{ implies } a - j > \alpha,$$

where the convex hull of $\text{sing supp}(S)$ is $[0, \alpha]$, and in (1), b can be taken to be $a - \alpha$.

We now turn to the case of several variables. Let $\Gamma \subset \mathbf{R}^n$ be a closed convex cone whose vertex is the origin, and which does not contain any straight line. We denote by $\mathcal{D}'(\Gamma)$ the space of distributions supported in Γ , and by \mathcal{D}'_Γ the space of distributions with support contained in some translate of Γ . It is well known [5] that if $S \in \mathcal{E}'$ has support contained in Γ , then $S * \mathcal{D}'(\Gamma) \subseteq \mathcal{D}'(\Gamma)$. A complete characterization of those $S \in \mathcal{E}'$ for which $S * \mathcal{D}'(\Gamma) = \mathcal{D}'(\Gamma)$ has been given by Hörmander [4], in Theorems 16.7.3, 16.7.4; in particular the condition asks that for every closed cone $\Gamma_1 \subset \Gamma$, there are positive constants C, M, A such that

$$(4) \quad |1/\hat{S}(z)| < C(1 + |z|)^M \exp(A|\text{Im } z|)$$

if $\text{Im } z \in \Gamma_1$ and $|\text{Im } z| > C \log(2 + |z|)$.

At first sight, it does not seem obvious to verify that every invertible distribution $S \in \mathcal{E}'(\mathbf{R})$ for which $\mathcal{H}(S)$ consists only of one function satisfies (4), but this is indeed an immediate consequence of Theorem 1 and Hörmander's results.

The result of [8], Theorem 1, can be extended in an obvious way to the case of distributions supported in cones; the proof uses the same arguments as in [8], and the Lion-Titchmarsh theorems for supports of convolutions, [5]: for this reason it

does not seem necessary to repeat it here; in this case the surjectivity condition is that S must be invertible, and that

$$(5) \quad \text{for all } a \in \mathbf{R}^n \text{ and all } u \in \mathcal{D}'_\Gamma, \text{sing supp}(S * u) \subseteq \Gamma + \{a\} \text{ implies} \\ \text{sing supp}(u) \subseteq \Gamma + \{a\}.$$

However, it is easy to show that, in (5), one cannot replace \mathcal{D}'_Γ with \mathcal{E}' . To prove it, one simply considers any partial differential operator $P(D)$ which is not hyperbolic with respect to Γ . Then it is obvious that condition (5) with \mathcal{D}'_Γ replaced by \mathcal{E}' is satisfied, while (5) itself is not.

2. In this section Γ will be an open convex cone, with vertex at the origin, and such that it does not contain any straight line; denote by Γ^* its dual cone $\Gamma^* = \{\xi \in \mathbf{R}^n : \langle \xi, x \rangle \geq 0 \text{ for all } x \in \Gamma\}$, and let $\mathcal{S}'(\Gamma^*)$ be the space of tempered distributions with supports contained in the closed cone Γ^* . Consider $S_1, \dots, S_m \in \mathcal{S}'(\Gamma^*)$: we look for necessary and sufficient conditions on S_1, \dots, S_m in order to ensure the existence of $E_1, \dots, E_m \in \mathcal{S}'(\Gamma^*)$ such that

$$(6) \quad S_1 * E_1 + \dots + S_m * E_m = \delta.$$

It is well known [9] that the Laplace transform is an isomorphism between $\mathcal{S}'(\Gamma^*)$ and the space $H(\Gamma)$ of functions f holomorphic in $\mathbf{R}^n + i\Gamma$ for which there are positive constants α, β, M such that

$$|f(z)| \leq M(1 + |z|^2)^\alpha(1 + \Delta^{-\beta}(\text{Im } z)), \quad \text{for all } z \in \mathbf{R}^n + i\Gamma,$$

where $\Delta(y) = \text{dist}(y, \partial\Gamma)$.

THEOREM 2. *Given $S_1, \dots, S_m \in \mathcal{S}'(\Gamma^*)$, a necessary and sufficient condition for the existence of E_1, \dots, E_m in $\mathcal{S}'(\Gamma^*)$ such that (6) holds, is that there exists positive constants C, γ, δ such that, for all $z \in \mathbf{R}^n + i\Gamma$,*

$$(7) \quad |\hat{S}_1(z)| + \dots + |\hat{S}_m(z)| \geq C(1 + |z|^2)^{-\gamma}(1 + \Delta^{-\delta}(\text{Im } z))^{-1}.$$

PROOF. By taking the Laplace transform of (6), the necessity of (7) is obvious. We only sketch the proof of its sufficiency, since it follows the well-known lines of [3]. First notice that even if $H(\Gamma)$ is not an A_p space in the sense of [3], it does satisfy the properties which are necessary to apply the $\bar{\partial}$ -techniques; indeed, since Γ is convex, the function $\log(1 + \Delta^{-\beta}(\text{Im } z))$ is plurisubharmonic; on the other hand, using the Laplace transform, one immediately sees that if $f \in H(\Gamma)$ its derivatives also belong to $H(\Gamma)$. In order to conclude, as in [3], with the construction of a suitable Koszul complex, it is sufficient to show that if an analytic function f satisfies an L^2 -estimate of the form

$$\int_{\mathbf{R}^n + i\Gamma} |f(z)|^2(1 + |z|^2)^{-\gamma}(1 + \Delta^{-\delta}(\text{Im } z))^{-1} d\lambda < +\infty,$$

where $d\lambda$ is the Lebesgue measure, then $f \in H(\Gamma)$. To prove this, it is sufficient to apply the mean value theorem to f on a ball centered at z and of radius $\Delta(\text{Im } z)/2$, and then to use the Cauchy-Schwarz inequality.

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