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ON THE SUPPORT OF QUASI-INVARIANT MEASURES ON INFINITE-DIMENSIONAL GRASSSMANN MANIFOLDS

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ABSTRACT. One antisymmetric analogue of Gaussian measure on a Hilbert space is a certain measure on an infinite-dimensional Grassmann manifold. The main purpose of this paper is to show that the characteristic function of this measure is continuous in a weighted norm for graph coordinates. As a consequence the measure is supported on a thickened Grassmann manifold. The action of certain unitary transformations, in particular smooth loops $S^1 \to U(n, \mathbf{C})$, extends to this thickened Grassmannian, and the measure is quasi-invariant with respect to these point transformations.

1. Introduction. Let H denote a separable Hilbert space with a given orthogonal decomposition $H = H_+ + H_-$, and let Gr denote the Hilbert-Schmidt Grassmannian of subspaces W of H which are close to H_+ in the following sense: (i) the orthogonal projection $Q_-: W \to H_-$ is Hilbert-Schmidt and (ii) the orthogonal projection $Q_+: W \to H_+$ is Fredholm of index zero. The graph coordinate at H_+ is the inverse of the mapping $\mathcal{L}_2(H_+, H_-) \to \mathrm{Gr}: Z \to \mathrm{graph}(Z)$ ($\mathcal{L}_2 =$ Hilbert-Schmidt). The manifold Gr is a homogeneous space for the restricted unitary group, which consists of all unitary transformations of $H = H_+ + H_-$ having the matrix form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with b and c Hilbert-Schmidt, index$(a) = 0$ (see [**2** or **3**]).

In [**2**] a family of finitely additive measures on Gr, $\{\mu_s : s > -1\}$, was constructed. It was shown that (for $s > 0$ and $\neq$ even integer) these measures are quasi-invariant with respect to the restricted unitary transformations in the sense of Irving Segal's algebraic integration theory (i.e., as set transformations). Also explicit integral expressions for the characteristic functions Φ_s were obtained, where the integration is with respect to infinite product measures $d\nu_s$ (of intrinsic interest) on a group of upper triangular matrices, S. These integral formulas were used to obtain estimates for the continuity of Φ_s (and hence the support of μ_s). The results of this paper will show that these estimates were quite crude.

The key result of this paper concerns the infinite product measures $d\nu_s$ on the group S. In (2.3) below we will show that the entries of the inverse matrix, x_{pq}^{-1}, as random variables on S, have distributions which are uniformly bounded by some constant with probability one. In [**2**] we showed that x^{-1} is an unbounded operator with probability one. Thus this result seems reasonably sharp.

This result will then be applied in (3.2) to show that Φ_s is continuous on a subspace of $\mathcal{L}_2(H_+, H_-)$ with a norm which is stronger than the given Hilbert-Schmidt norm. It seems likely to me that Φ_s is continuous on $\mathcal{L}_2(H_+, H_-)$, but proving this will require a more subtle argument. The continuity of Φ_s implies that

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μ_s is supported on a thickened Grassmannian. A large subgroup of the restricted unitary transformations (containing smooth loops) acts continuously on the thickened Grassmannian, and the μ_s ($s \geq 0$) are quasi-invariant with respect to these point transformations. This is shown in §3.

Notation. $dm(\cdot)$ denotes Lebesgue measure.

2. The infinite product measures on the upper triangular group S. Let $\{e_j : j > 0\}$ be an orthonormal basis for the Hilbert space H_+. Let S denote the set of all upper triangular matrices $x = (x_{pq})_{0<p\leq q<\infty}$ with $x_q = x_{qq} > 0$ and $x_{pq} \in \mathbf{C}$ for $p < q$. S is a group with respect to composition of operators (on H_+^{alg}, the algebraic span of the e_j).

We define a probability measure on S (for each $s > -1$) by

$$d\nu_s(x) = \prod_{0<p<q<\infty} \pi^{-1} e^{-|x_{pq}|^2} dm(x_{pq}) \prod_{0<q<\infty} 2x_q e^{-x_q^2} x_q^{2(q+s-1)} \frac{dx_q}{\Gamma(q+s)}.$$

We will think of the x_{pq} as random variables on S, and write Prob$(\ldots)$ for the probability of an event relative to ν_s.

Let $\xi_{pq} = x_{pq}/\exp(x_{qq}^2)$ for $p < q$.

(2.1) LEMMA. $\operatorname{Prob}(\overline{\lim}\, q^N |\xi_{pq}| = 0) = 1$ *for any N, where the limit is over all $p < q$, both p and q tending to infinity.*

PROOF. $|x_{pq}|$ has the density e^{-t}, while x_{qq}^2 has the density $\Gamma(q+s)^{-1} e^{-u} u^{q+s-1}$. Hence

$$\begin{aligned}
&\operatorname{Prob}(|x_{pq}| > \varepsilon \exp(x_{qq}^2)) \\
&= \frac{1}{\Gamma(q+s)} \int_0^\infty e^{-u} u^{q+s-1} \left\{ \int_{\varepsilon \exp(u)}^\infty e^{-t}\, dt \right\} du \\
&= \Gamma(q+s)^{-1} \int_0^\infty e^{-u} u^{q+s-1} e^{-\varepsilon \exp(u)}\, du.
\end{aligned}$$

Fix q and let $G(\varepsilon)$ denote the above probability. G is a continuous function of ε for $\varepsilon \geq 0$, $G(0) = 1$, and for $\varepsilon > 0$ it is differentiable with

$$G'(\varepsilon) = -\Gamma(q+s)^{-1} \int_0^\infty u^{q+s-1} e^{-\varepsilon \exp(u)}\, du.$$

This tends to $-\infty$ as $\varepsilon \downarrow 0$. Thus for any N, provided ε is sufficiently small,

$$G(\varepsilon) < e^{-\varepsilon q^{N+1}}. \tag{2.2}$$

To apply the first Borel-Cantelli lemma, we consider

$$\begin{aligned}
&\sum_{0<p<q<\infty} \operatorname{Prob}(|\xi_{pq}| > c q^{-N}) \\
&\quad = \sum_{1<q<\infty} (q-1)\Gamma(q+s)^{-1} \int_0^\infty e^{-u} u^{q+s-1} e^{-(c/q^N)e^u}\, du.
\end{aligned}$$

By (1.2) (with $\varepsilon = c/q^N$) this sum is bounded by

$$\sum_{1<q<\infty} (q-1) e^{-cq} < \infty.$$

Thus Borel-Cantelli implies that $\operatorname{Prob}(\overline{\lim}\, q^N |\xi_{pq}| > c) = 0$. Let $c \downarrow 0$ to obtain (2.1). $\square$

(2.3) THEOREM. $\operatorname{Prob}(\bigcup_{c>0}\{|x_{pq}^{-1}| < c \text{ for all } p < q\}) = 1$.

PROOF. $x = (1+\xi)\operatorname{diag}(\exp(x_{qq}^2))$. Thus

$$(x^{-1})_{pq} = \frac{1}{\exp(x_{pp}^2)} \sum (-1)^k \xi_{pj_1} \cdots \xi_{j_{k-1}q}$$

where the sum is over all tuples of integers $(j_1, \ldots, j_k)$ with $p < j_1 < \cdots < j_k = q$.

Now suppose $|x_{pp}^{-1}| < A$ for all p, $|\xi_{pq}| \le A$ for all $p < q \le N$ and $|\xi_{pq}| < q^{-2}$ for $q > N$, where $1 < A$. Then

$$\begin{aligned} |(x^{-1})_{pq}| &\le \exp(A^2) \sum |\xi_{pj_1}| \cdots |\xi_{j_{k-1}q}| \\ &\le \exp(A^2) A^N \sum j_1^{-2} \ldots j_k^{-2} \\ &= \exp(A^2) A^N \prod_{p<j\le q} (1 + j^{-2}). \end{aligned} \tag{2.4}$$

This is bounded by a constant not depending upon (p, q).

The lemma (1.1) implies that both $\operatorname{Prob}(\bigcup_N \{|\xi_{pq}| < q^{-2} \text{ for } q > N\})$ and $\operatorname{Prob}(\bigcup_A \{|\xi_{pq}| < A \text{ for all } p < q\})$ equal 1, while

$$\operatorname{Prob}(\{|x_{pp}^{-2}| < A \text{ for all } p\}) = \prod_1^\infty \int_{A^{-2}}^\infty ue^{-u} u^{q+s-1} \frac{du}{\Gamma(q+s)} \to 1$$

as $A \to \infty$ (see §12.2 of [**1**]). Thus (2.4) implies (2.3). □

In [**2**] it was shown that $\operatorname{Prob}(\{|x^{-1}|_\infty < \infty\}) = 0$ (here $|x^{-1}|_\infty$ is the sup norm relative to the H_+ norm for x^{-1} acting on H_+^{alg}).

3. Continuity of Φ_s and the smooth Grassmannian. Let $\{e_j : j \in Z, j \neq 0\}$ be an orthonormal basis for H with $e_j \in H_\pm$ when $\pm j > 0$, so that we can identify operators from H_+ to H_- with matrices $(Z_{ij})_{i<0,j>0}$. Let M denote the space of all such matrices. For each $s > -1$ we can define a (following [**2**]) probability measure μ_s on M by either of the following two methods: (1) we can specify the push-down of μ_s under the projection

$$p: M \to \mathcal{L}(\mathbf{C}^n, \mathbf{C}^m): (Z_{ij})_{-m\le i<0, n>j>0},$$

namely

$$p_* \mu_s = c \det(1 + Z^* Z)^{-m-n-s} \, dm(Z)$$

(c is a normalization constant); or (2) we can specify the characteristic function of μ_s, namely

$$\Phi_s(Z) = \int e^{i \operatorname{Re} \operatorname{tr} Z^* W} d\mu_s(W) = \int e^{-\operatorname{tr} Z^* Z x^{-1*} x^{-1}} \, d\nu_s(x), \tag{3.1}$$

where $d\nu_s$ is the measure on S defined in §2 (here $Z \in M$ is a matrix with $Z_{ij} = 0$ for all but finitely many (i, j)).

As mentioned in the introduction, it seems likely to me that Φ_s extends to a continuous function on $\mathcal{L}_2(H_+, H_-)$; for this to be true, it is necessary and sufficient to show that trace $Z^* Z x^{-1^*} x^{-1} \to 0$ in probability (relative to $d\nu_s$) as $\operatorname{tr} Z^* Z \to 0$. At present I can only prove the followig weaker result.

(3.2) THEOREM. *Φ_s is continuous relative to the norm $\|\ \|$ on finite rank matrices given by $\|Z\| = \sum_{j>0} |j|^{1/2}|Z(e_j)|$.*

PROOF.

$$|(x^{-1*}x^{-1})_{nm}| \leq \sum_{1\leq k\leq \min(n,m)} |x_{kn}^{-1}||x_{km}^{-1}| \leq C\min(n,m)$$

for all $n > 0$, $m > 0$ with probability $\uparrow 1$ as $C \uparrow \infty$, by (2.3). Also

$$\begin{aligned}|\operatorname{trace} Z^*Zx^{-1*}x^{-1}| &= \left|\sum_{n>0,m>0} (Z^*Z)_{nm}(x^{-1*}x^{-1})_{mn}\right| \\ &\leq \left(\sum_{n,m} \min(n,m)|Z^*Z|_{nm}\right) \sup_{n,m} \frac{1}{\min(n,m)}|(x^{-1*}x^{-1})_{nm}|,\end{aligned}$$

and $\sum_{n,m} \min(n,m)|Z^*Z|_{nm} \leq \|Z\|^2$. Thus pointwise, with probability one relative to $d\nu_s$, $\operatorname{tr} Z^*Zx^{-1*}x^{-1} \to 0$ as $\|Z\| \to 0$. Lebesgue dominated convergence applied to (3.1) now implies (3.2). $\square$

We now briefly describe the relevance of this result to the theory of loop groups. Suppose $H = L^2(S^1, \mathbf{C})$ and H_+ is the usual Hardy subspace of boundary values of holomorphic functions in the disk, and let $LU(n)$ denote the group of smooth maps $S^1 \to U(n, \mathbf{C})$. The group $LU(n)$ acts on $L^2(S^1, \mathbf{C}^n)$ unitarily via the pointwise action of $U(n)$ on $\mathbf{C}^n$. Hence we can view $LU(n)$ as a subgroup of $U(H)$ if we identify $L^2(S^1, \mathbf{C}^n)$ with H via $\varepsilon_i z^j \to z^{i+jn}$, where $\{\varepsilon_i : 0 \leq i < n\}$ is the usual basis for $\mathbf{C}^n$. For a loop $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ as an operator on $H_+ + H_-$, a is the block Toeplitz operator and b the block Hankel operator defined by γ. We will show that loops act on a certain completion of Gr, that this completion supports the measures μ_s, and that the μ_s are quasi-invariant with respect to this action.

Let H^t denote the Sobolev space of distributions ϕ on S^1 with $\sum(1+k^2)^{t/2}|\hat{\phi}(k)|^2 < \infty$, H_+^t the subspace of distributions with $\hat{\phi}(k) = 0$ for $k < 0$, and H_-^t the orthogonal complement. Note $\|Z\| \leq c(\varepsilon)|Z|_\varepsilon$, $|Z|_\varepsilon^2 = \sum_{j>0} |j|^{2+\varepsilon}|Z(e_j)|^2$, $\varepsilon > 0$. For all t sufficiently large the inclusion of $L_2(H_+^t, H_-^t)$ into the space of Z's with norm $|\cdot|_\varepsilon$ is Hilbert-Schmidt. Fix such a t. Let

$$E = H_+^t + H_-^t \ (= E_+ + E_-) \tag{3.3}$$

and define the Grassmannian $\operatorname{Gr}(E)$ corresponding to the decomposition (3.3) as in the introduction. Minlos's theorem (see Theorem 3.1 of [**5**]) and (3.2) imply that the measures μ_s $(s > -1)$ are supported on the graph coordinate system of E_+, which is modelled on those (Z_{ij}) satisfying $\sum(1+i^2)^{-t/2}(1+j^2)^{-t/2}|Z_{ij}|^2 < \infty$.

There is a natural mapping of graph coordinate systems $\tau : \mathcal{L}_2(H_+, H_-) \to \mathcal{L}_2(E_+, E_-)$. In terms of subspaces, τ maps $W \in \operatorname{Gr}$ to $W \cap E \in \operatorname{Gr}(E)$. To see that τ is globally well defined and continuous on Gr, it suffices to check graph coordinate neighborhoods of the points $H_S \in \operatorname{Gr}$ (respectively, $E_S \in \operatorname{Gr}(E)$) defined in §2 of [**3**]; τ is again the natural map $\mathcal{L}_2(H_S, H_S^\perp) \to \mathcal{L}_2(E_S, E_S^\perp)$.

We now consider two subgroups of the restricted unitary group of H, U_{res}. The first is K, which consists of those unitary transformations of H of the form $g = \left(\begin{smallmatrix} a & \\ & d\end{smallmatrix}\right)$, where g maps H^t homeomorphically onto itself (K is two copies of the unitary subgroup in [**5**]). The elements of K act on E (restrict a to E_+, extend d continuously to E_-), hence on $\mathrm{Gr}(E)$.

For the second, note that the set of linear operators ξ on H which extend to continuous maps $H^{-t} \to H^t$ is a Banach algebra with respect to the operator norm of $\xi: H^{-t} \to H^t$. It follows that the unitary operators on H of the form $1+\xi$, with ξ as above, form a Banach Lie group N (this is a slight variant of an example in [**6**, p. 0.3]). Note N maps continuously into U_{res}, and N contains $U(\infty)$ as a dense subgroup ($U(\infty)$ is defined relative to $\{z^j\}$). To see that the elements of N extend to continuous transformations of $\mathrm{Gr}(E)$, it seems best to consider the action in graph coordinates relative to the $H_S: Z \to (C+DZ)(A+BA)^{-1}$ (here $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ relative to $H = H_S + H_S^{\perp}$). This shows the action of g extends locally to a neighborhood of any $W \in \mathrm{Gr}(E)$; since the same is true for g^{-1} and Gr is dense in $\mathrm{Gr}(E)$, these are globally well defined, continuous invertible transformations.

Now suppose $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a loop in the identity component of $LU(n)$, $LU(n)_0$. The operators a and d are Fredholm of index zero, so they have polar decomposition $a = q_1|a|$, $d = q_2|d|$. The factors of the decomposition

$$g = \begin{pmatrix} q_1 & \\ & q_2 \end{pmatrix} \begin{pmatrix} |a| & q_1^{-1}b \\ q_2^{-1}c & |d| \end{pmatrix}$$

are in K and N respectively. For the smoothness of g easily implies that $\left(\begin{smallmatrix} & b \\ c & \end{smallmatrix}\right): H^{-t} \to H^t$ is bounded, $\left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) = g - \left(\begin{smallmatrix} & b \\ c & \end{smallmatrix}\right): H^t \to H^t$ is Fredholm, and the kernels and cokernels of a and d consist of smooth functions. Hence $|a| = \sqrt{1-b^*b}$ is of the form $1+\xi$, $\xi: H^{-t} \to H^t$ continuous, and $q: H^t \to H^t$ is a homeomorphism.

With these preliminaries aside, we now prove

(3.4) THEOREM. *For $s \geq 0$, μ_s is quasi-invariant with respect to the point transformations on* $\mathrm{Gr}(E)$ *of the group KN; in particular, μ_s is quasi-invariant with respect to the action of loops in $LU(n)_0$. For $g \in KN$, the Radon-Nikodým derivative is* $\det|a(g^{-1}) + b(g^{-1})z|^{2s}$ *(note this operator has a determinant, because* $|a|^2 = 1 - |c|^2$).

PROOF. The elements of K act linearly on graph coordinates at $H_+: Z \to dZa^{-1}$. Using the unitarity of $g = \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) \in K$, it is easily checked that the characteristic function of $g_*\mu_s$ is Φ_s. Hence μ_s is K-invariant.

For $g \in U(\infty)$, because g maps cylinder sets to cylinder sets, it's easily checked that $g_*\mu_s = \rho(g,\cdot)^s\, d\mu_s$, where $\rho(g,Z) = \det|a(g^{-1}) + b(g^{-1})Z|^2$ (see §2 of [**2**]). If $g \in N$, we can find $g_n \in N$ such that $g_n \to g$ in N. If ϕ is a bounded continuous function on $\mathrm{Gr}(E)$,

$$\int \phi(g_n \cdot z)\, d\mu_s(z) = \int \phi(z)\rho(g_n, z)^s\, d\mu_s(z). \tag{3.5}$$

By Lemma (4.4) of [**2**] $\rho(g_n,z)^s$ converges in $L^1(\mu_s)$ to a function $\rho(g,z)^s$, which must equal $\det|a(g^{-1}) + b(g^{-1})z|^{2s}$, because this is the pointwise limit. We now take the limit $n \to \infty$ in (3.5); we apply dominated convergence on the left to conclude

$$\int \phi d g_*\mu_s = \int \phi\rho(g,\cdot)^s\, d\mu_s,$$

for all bounded continuous ϕ on $\mathrm{Gr}(E)$. This implies (3.4). □

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