RESTRICTED LEFT INVERTIBLE TOEPLITZ OPERATORS  
ON MULTIPPLY CONNECTED DOMAINS  
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Dedicated to Professor Junzo Wada on his sixtieth birthday  

ABSTRACT. A characterization of restricted left invertible Toeplitz operators on multiply connected domains is given. To prove this, some extension theorems are given.

1. Introduction. Let \( \Omega \) be a bounded connected open subset of the plane whose boundary \( \Gamma \) consists of finitely many disjoint analytic Jordan curves. We denote by \( H^\infty(\Omega) \) the space of bounded holomorphic functions on \( \Omega \). In [1], Abrahamse studied Toeplitz operators on \( \Omega \) and gave a left invertibility criterion for Toeplitz operators in a generalized sense. In [3], Clancey and Gosselin studied local properties of Toeplitz operators on the unit open disk and gave a characterization of restricted left invertible Toeplitz operators with the help of Younis’ extension theorem [7].

In this paper, we will extend the Clancey-Gosselin-Younis theorem to multiply connected domains (§3). To prove this, we need some extension theorems as in [7, 8]. Since \( H^\infty(\Omega) \) is not a strongly logmodular algebra, here we cannot use the Younis extension theorems. We will give more general extension theorems (§2).

2. Extension theorems. Let \( \mathcal{A} \) be a function algebra on \( X \). Throughout this section, we assume that \( X \) is a totally disconnected compact Hausdorff space. A closed subset \( E \) of \( X \) will be called a peak set for \( \mathcal{A} \) if there is a function \( f \) in \( \mathcal{A} \), which is called a peaking function for \( E \), such that \( f = 1 \) on \( E \) and \( |f| < 1 \) on \( X \setminus E \). A subset \( S \) of \( X \) is called a weak peak set for \( \mathcal{A} \) if it is an intersection of some peak sets. We consider the following separation condition.

\[
\# \quad \text{For any open and closed subset } U \text{ of } X, \text{ there is a function } G \text{ in } \mathcal{A} \text{ such that } |G| > 1 \text{ on } U \text{ and } |G| < 1 \text{ on } X \setminus U.
\]

The following theorem is a generalization of [8, Theorem 3.2 and Corollary 3.3].

**Theorem 1.** Suppose that \( \mathcal{A} \) satisfies separation condition \( \# \). Let \( S \) be a weak peak set for \( \mathcal{A} \). If a function \( f \) in \( \mathcal{A} \) satisfies \( |f| > 0 \) on \( S \), then there exists a function \( F \) in \( \mathcal{A} \) such that \( F = f \) on \( S \) and \( |F| > 0 \) on \( X \).

**Proof.** We may assume that \( f \) vanishes somewhere on \( X \). Take an open and closed subset \( U \) of \( X \) such that \( S \subset U \), \( |f| > 0 \) on \( U \), and

\[
|f| \leq 1 \quad \text{on } X \setminus U.
\]

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Put $c = \inf\{|f(x)|; x \in U\}$. Then $c > 0$. Since $S$ is a weak peak set, there is a peak set $E$ for $A$ with $S \subset E \subset U$ \cite[p. 56]{5}. Let $h \in A$ be a peaking function for $E$. Replacing $h$ by high powers of $h$, we may assume
\begin{equation}
|h| \leq 1/2 \quad \text{on } X \setminus U.
\end{equation}
By our separation condition, there is a function $G$ in $A$ such that
\begin{equation}
|G| \leq c/3 \quad \text{on } U \quad \text{and} \quad |G| \geq 3 \quad \text{on } X \setminus U.
\end{equation}
Put $F = f + G(1 - h)$. Clearly $F \in A$ and $F = f$ on $S$. For $x \in U$, by (3) we have
\begin{equation}
|F(x)| \geq |f(x)| - |G(x)||1 - h(x)| \geq c/3 > 0.
\end{equation}
For $x \in X \setminus U$, by (1), (2), and (3) we have
\begin{equation}
|F(x)| \geq |G(x)||1 - h(x)| - |f(x)| \geq 1/2 > 0.
\end{equation}
Thus $|F| > 0$ on $X$.

**REMARK 1.** Let $A$ be a strongly logmodular algebra on a totally disconnected compact space $X$, that is, $\log |A^{-1}| = C_R(X)$, where $A^{-1}$ denotes the set of invertible elements in $A$ and $C_R(X)$ denotes the space of real continuous functions on $X$. Then $A$ satisfies $\langle \# \rangle$.

**REMARK 2.** For a general function algebra $A$, the assertion of Theorem $1$ is not true. Put $X = \{(z, t); z$ is a complex number and $t$ is a real number with $|z| = 1 - |t|$ and $|t| \leq 1\}$. Let $A = \{g \in C(X); \text{for each } t \text{ with } |t| < 1, g(z, t) \text{ has a continuous holomorphic extension to } \{|z| \leq 1 - |t|\}\}$. Then $S = \{(z, 0); |z| = 1\}$ is a peak set for $A$. Put $f(z, t) = (1 - |t|)z$. Then $f \in A$ and $|f| > 0$ on $S$, but it is easy to see that there is no $F \in A$ with $F = f$ on $S$ and $|F| > 0$ on $X$.

For a weak peak set $S$, put $A_S = \{f \in C(X); f|S \in A|S\}$. Then $A_S$ is a closed subalgebra of $C(X)$. For $\psi \in C(X)$ and $B \subset C(X)$, put $d(\psi, B) = \inf\{||\psi - f||; f \in B\}$. For $\psi \in C(X)$ and $F \subset X$, put $\|\psi\|_F = \sup\{|\psi(x)|; x \in F\}$. The following theorem is a generalization of \cite[Theorem 3.1]{8}.

**THEOREM 2.** Suppose that $A$ satisfies separation condition $\langle \# \rangle$. Let $S$ be a weak peak set for $A$, and let $u$ be a function in $C(X)$ such that $\|u\| = 1$, $|u| = 1$ on $S$, and $d(u, A_S) < 1$. Then there is a function $\tilde{u}$ in $C(X)$ such that $|\tilde{u}| = |u|$ on $X$, $\tilde{u} = u$ on $S$, and $d(\tilde{u}, A) < 1$.

**PROOF.** Since $d(u, A_S) < 1$, there exists a function $f$ in $A$ such that $\|u - f\|_S < 1$. Since $|u| = 1$ on $S$, $|f| > 0$ on $S$. By Theorem 1, there is a function $F$ in $A$ such that $F = f$ on $S$, and
\begin{equation}
|F| > 0 \quad \text{on } X.
\end{equation}
Since $\|u - F\|_S = \|u - f\|_S < 1$, there is an open and closed subset $U$ of $X$ such that $S \subset U$ and
\begin{equation}
\|u - F\|_U < 1.
\end{equation}
Since $S$ is a weak peak set, there is a peak set $E$ for $A$ such that $S \subset E \subset U$. Let $h \in A$ be a peaking function for $E$. Replacing $h$ by high powers of $(1 + h)/2$, we may assume that
\begin{equation}
|h| > 0 \quad \text{on } X \quad \text{and} \quad 0 < |hF| < 1 \quad \text{on } X \setminus U.
\end{equation}
Put

\[ \tilde{u} = \begin{cases} \frac{uh}{|h|} & \text{on } U, \\ |u|hF/|hF| & \text{on } X \setminus U. \end{cases} \]

By (3), \( \tilde{u} \in C(X) \). Clearly \( |\tilde{u}| = |u| \) on \( X \) and \( \tilde{u} = u \) on \( S \). To prove \( d(\tilde{u}, A) < 1 \), it is sufficient to prove \( \|\tilde{u} - hF\|_{X \setminus U} < 1 \), because \( hF \in A \). We have

\[
\|\tilde{u} - hF\|_{X \setminus U} = \left\| \frac{|u|hF}{|hF|} - \frac{hF}{|hF|} \right\|_{X \setminus U} = \|u| - |hF|\|_{X \setminus U} < 1 \quad \text{by (3) and } \|u\| \leq 1.
\]

Also we have

\[
\|\tilde{u} - hF\|_U = \left\| \frac{uh}{|h|} - \frac{hF}{|h|} \right\|_U = \|u - F|h|\|_U.
\]

Let \( x \in U \). Then

\[
|u(x) - F(x)|h(x)| = |(1 - |h(x)|)u(x) + |h(x)|(|u(x) - F(x)|)\|
\leq 1 - |h(x)| + |h(x)|\|u - F\|_U \quad \text{by } \|u\| \leq 1, \|h\| \leq 1
\]

\[
= 1 - |h(x)|\{1 - \|u - F\|_U\}
\]

Thus we get \( \|\tilde{u} - hF\|_U < 1 \), hence \( \|\tilde{u} - hF\| < 1 \).

If \( A \) is a strongly logmodular algebra, then \( A_S \), where \( S \) is a weak peak set for \( A \), is generated by \( A \) and \( \{f^{-1}; f \in A \cap A_S^{-1}\} \) [2]. Younis used this property to prove his theorem [8, Theorem 3.1]. So the following theorem is another generalization of [8, Theorem 3.1].

**THEOREM 3.** Suppose that \( A \) satisfies separation condition (\#). Let \( S \) be a weak peak set for \( A \) such that \( A_S \) is generated by \( A \) and \( \{f^{-1}; f \in A \cap A_S^{-1}\} \). If \( u \) is a function in \( C(X) \) with \( \|u\| \leq 1 \) and \( d(u, A_S) < 1 \), then there exists a function \( \tilde{u} \in C(X) \) such that \( |\tilde{u}| = |u| \) on \( X \), \( \tilde{u} = u \) on \( S \), and \( d(\tilde{u}, A) < 1 \).

**NOTE.** We do not assume \( |u| = 1 \) on \( S \).

**PROOF.** By our assumption, there exist functions \( g \) in \( A \) and \( f \) in \( A \cap A_S^{-1} \) such that

\[
(1) \quad \|u - f^{-1}g\| < 1.
\]

Since \( f^{-1} \in A_S \), \( |f^{-1}| > 0 \) on \( X \). By Theorem 1, there is a function \( F \) in \( A \) such that \( F = f^{-1} \) on \( S \) and \( |F| > 0 \) on \( X \). Put \( \tilde{u} = ufF/|fF| \). Since \( |fF| > 0 \) on \( X \), \( \tilde{u} \in C(X) \). Clearly \( |\tilde{u}| = |u| \) on \( X \) and \( \tilde{u} = u \) on \( S \). To prove \( d(\tilde{u}, A) < 1 \), suppose not. Since \( \|\tilde{u}\| = \|u\| \leq 1 \), we have \( d(\tilde{u}, A) = 1 \). We note that the space of bounded linear functionals of \( C(X)/A \) may be identified with \( A^\perp \), the set of regular Borel measures \( \mu \) on \( X \) such that \( \int_X \phi d\mu = 0 \) for every \( \phi \in A \). Hence there exists \( \mu \in A^\perp \) with the unit total variation, \( \|\mu\| = 1 \), such that

\[
1 = \int_X \tilde{u} d\mu = \int_X \frac{ufF}{|fF|} d\mu.
\]

Since \( \|u\| \leq 1 \), we get \( ((ufF)/|fF|)\mu = |\mu| \), hence

\[
(2) \quad ufF\mu = |fF||\mu|.
\]
Then

\[ 1 > \left| \int_X (u - f^{-1}g) \frac{fF}{\|fF\mu\|} \, d\mu \right| \quad \text{by (1)} \]

\[ = \frac{1}{\|fF\mu\|} \left| \int_X uF \, d\mu \right| \quad \text{by } \mu \in A^\perp \]

\[ = \frac{1}{\|fF\mu\|} \int_X |fF| \, d|\mu| \quad \text{by (2)} \]

\[ = 1. \]

This contradiction shows \( d(\tilde{u}, A) < 1 \).

3. Restricted left invertible Toeplitz operators. Let \( \Omega \) be a bounded connected open subset of the plane whose boundary \( \Gamma \) consists of finitely many disjoint analytic Jordan curves. Identifying a function in \( H^\infty(\Omega) \) with its boundary function, we may regard \( H^\infty(\Omega) \) as an essentially supremum norm closed subalgebra of \( L^\infty(m) \), where \( m \) is the arc length measure on \( \Gamma \). A closed subspace \( M \) of \( L^2(m) \) with \( M \neq \{0\} \) is called invariant if \( H^\infty(\Omega)M \subseteq M \), and it is called reducing if \( H^\infty(\Omega)M \subset M \) and \( \overline{H^\infty(\Omega)}M \subset M \). An invariant subspace of \( L^2(m) \) is called simple if it contains no reducing subspaces. For a closed subspace \( M \), \( P_M \) denotes the orthogonal projection of \( L^2(m) \) onto \( M \). For a function \( \psi \) in \( L^\infty(m) \), put \( T^M_\psi(f) = P_M(\psi f) \) for every \( f \) in \( M \). \( T_\psi = \{T^M_\psi; M \text{ is a simply invariant subspace}\} \) is called a family of Toeplitz operators with a symbol \( \psi \). In the case that \( \Omega \) is the unit open disk, by Beurling’s theorem, \( T^M_\psi \) is unitarily equivalent to the usual Toeplitz operator \( T_\psi \). We call \( T_\psi \) left invertible if \( T^M_\psi \) is a left invertible operator on \( M \) for every simply invariant subspace \( M \). In [1, Theorem 4.1], Abrahamse proved the following theorem which is a generalization of the Devinatz-Rabindranathan theorem (see [6, p. 119]).

**Theorem 4.** Let \( \psi \) be a function in \( L^\infty(m) \) with \( |\psi| = 1 \) a.e. \( dm \). Then the following conditions are equivalent:

(i) \( T_\psi \) is left invertible.

(ii) \( d(\psi, H^\infty(\Omega)) < 1 \).

We note that by his proof, (i) \( \Rightarrow \) (ii) is true for every \( \psi \in L^\infty(m) \) with \( \|\psi\| \leq 1 \).

Let \( X \) be the maximal ideal space of \( L^\infty(m) \). Then \( X \) is a totally disconnected compact Hausdorff space [4, p. 190]. Identifying a function in \( L^\infty(m) \) with its Gelfand transform, we have \( L^\infty(m) = C(X) \). We may consider \( H^\infty(\Omega) \) as a function algebra on \( X \) [4, p. 123]. Let \( S \) be a weak peak subset of \( X \) for \( H^\infty(\Omega) \). A family of Toeplitz operators \( T_\psi, \psi \in C(X) \), is called \( S \)-restricted left invertible if there is a function \( \Psi \) in \( C(X) \) such that \( \Psi = \psi \) on \( S \) and \( T_\psi \) is left invertible. In the case that \( \Omega \) is the open unit disk, Clancey-Gosselin-Younis [3, 7] gave a characterization of \( S \)-restricted left invertible Toeplitz operators as follows: If \( |\psi| = 1 \) a.e. \( dm \), then the Toeplitz operator \( T_\psi \) is \( S \)-restricted left invertible if and only if \( d(\psi, H^\infty_S) < 1 \). We shall give a generalization of the above theorem to multiply connected domains as an application of §2.
THEOREM 5. Let $S$ be a weak peak subset of $X$ for $H^\infty(\Omega)$. Let $\psi$ be a function in $C(X)$ with $|\psi| = 1$ on $S$. Then the following conditions are equivalent:

(i) $T_\psi$ is $S$-restricted left invertible.

(ii) $d(\psi, H^\infty(\Omega)_S) < 1$.

PROOF. (ii) $\Rightarrow$ (i) Since $X$ is totally disconnected, we may assume that $|\psi| = 1$ on $X$. We note that $A = H^\infty(\Omega)$ satisfies separation condition $(\#)$ in §2 [5, p. 119]. Then $H^\infty(\Omega)$, $\psi$, and $S$ satisfy all assumption of Theorem 2. Hence there exists a function $\Psi$ in $C(X)$ such that $|\Psi| = 1$ on $X$, $\Psi = \psi$ on $S$, and $d(\Psi, H^\infty(\Omega)) < 1$. By Theorem 4, $T_\Psi$ is left invertible. So (i) holds.

(i) $\Rightarrow$ (ii) Suppose that $T_\psi$ is $S$-restricted left invertible. By our definition, there is a function $\Psi$ in $C(X)$ such that $\Psi = \psi$ on $S$ and $T_\psi$ is left invertible. First we shall show that there is a function $h$ in $H^\infty(\Omega)$ such that

(1)

$h = 1$ on $S$, $\|\Psi h\| \leq 1$, and $|h| > 0$ on $X$.

To prove this, put

(2)

$\phi = \max\{|\Psi|, 1\}$.

Then $\phi \in C(X)$, $\phi \geq 1$ on $X$, and $\phi = 1$ on $S$. By [5, p. 58], there is a function $g$ in $H^\infty(\Omega)$ such that $g = 1$ on $S$ and

(3)

$|g| \leq 1/\phi$ on $X$.

Then $\|g\| = 1$. Take a positive integer $N$ with

(4)

$N \geq 4$ and $\|\Psi\|(5/6)^N \leq 1$.

Put $h = ((g + 2)/3)^N$. Then $h \in H^\infty(\Omega)$, $h = 1$ on $S$, and $|h| > 0$ on $X$. If $x \in X$ with $|g(x)| \leq 1/2$, then

$$|\Psi(x)h(x)| \leq \|\Psi\|(5/6)^N \leq 1 \quad \text{by (4)}.$$  

If $x \in X$ with $1/2 \leq |g(x)| \leq 1$, then

$$|\Psi(x)h(x)| \leq ((|g(x)| + 2)/3)^N |\phi(x)| \quad \text{by (2)}$$

$$\leq ((|g(x)| + 2)/3)^N /|g(x)| \quad \text{by (3)}$$

$$\leq 1.$$  

The last inequality follows from $((t + 2)/3)^N \leq t$ for $1/2 \leq t \leq 1$ and $N \geq 4$. Thus $\|\Psi h\| \leq 1$ and we get (1).

We put $\Phi = \Psi h \in L^\infty(m)$. By (1),

(5)

$\|\Phi\| \leq 1$ and $\Phi = \psi$ on $S$.

Also we have that $T_\Phi$ is left invertible. To see this, let $M$ be a simply invariant subspace of $L^2(m)$. Since $h \in H^\infty(\Omega)$ and $|h| > 0$ on $X$, $T_h^M$ is left invertible. Since $T_\psi$ is left invertible, $T_\psi^M = T_\Psi^M T_h^M$ is left invertible, so $T_\Phi$ is left invertible. As noted after Theorem 4, we get $d(\Phi, H^\infty(\Omega)) < \|\Phi\| \leq 1$. Hence

$$d(\psi, H^\infty(\Omega)_S) = d(\Phi, H^\infty(\Omega)_S) \quad \text{by (5)}$$

$$\leq d(\Phi, H^\infty(\Omega)) < 1.$$  

This completes the proof.
References


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