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RESTRICTED LEFT INVERTIBLE TOEPLITZ OPERATORS ON MULTIPLY CONNECTED DOMAINS

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Dedicated to Professor Junzo Wada on his sixtieth birthday

ABSTRACT. A characterization of restricted left invertible Toeplitz operators on multiply connected domains is given. To prove this, some extension theorems are given.

1. Introduction. Let Ω be a bounded connected open subset of the plane whose boundary Γ consists of finitely many disjoint analytic Jordan curves. We denote by $H^\infty(\Omega)$ the space of bounded holomorphic functions on Ω. In [**1**], Abrahamse studied Toeplitz operators on Ω and gave a left invertibility criterion for Toeplitz operators in a generalized sense. In [**3**], Clancey and Gosselin studied local properties of Toeplitz operators on the unit open disk and gave a characterization of restricted left invertible Toeplitz operators with the help of Younis' extension theorem [**7**].

In this paper, we will extend the Clancey-Gosselin-Younis theorem to multiply connected domains (§3). To prove this, we need some extension theorems as in [**7**, **8**]. Since $H^\infty(\Omega)$ is not a strongly logmodular algebra, here we cannot use the Younis extension theorems. We will give more general extension theorems (§2).

2. Extension theorems. Let A be a function algebra on X. Throughout this section, we assume that X is a totally disconnected compact Hausdorff space. A closed subset E of X will be called a peak set for A if there is a function f in A, which is called a peaking function for E, such that $f = 1$ on E and $|f| < 1$ on $X\backslash E$. A subset S of X is called a weak peak set for A if it is an intersection of some peak sets. We consider the following separation condition.

(#) For any open and closed subset U of X, there is a function G in A such that $|G| > 1$ on U and $|G| < 1$ on $X\backslash U$.

The following theorem is a generalization of [**8**, Theorem 3.2 and Corollary 3.3].

THEOREM 1. *Suppose that A satisfies separation condition (#). Let S be a weak peak set for A. If a function f in A satisfies $|f| > 0$ on S, then there exists a function F in A such that $F = f$ on S and $|F| > 0$ on X.*

PROOF. We may assume that f vanishes somewhere on X. Take an open and closed subset U of X such that $S \subset U$, $|f| > 0$ on U, and

$$|f| \leq 1 \quad \text{on } X\backslash U. \tag{1}$$

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Put $c = \inf\{|f(x)|; x \in U\}$. Then $c > 0$. Since S is a weak peak set, there is a peak set E for A with $S \subset E \subset U$ [**5**, p. 56]. Let $h \in A$ be a peaking function for E. Replacing h by high powers of h, we may assume

$$|h| \leq 1/2 \quad \text{on } X \backslash U. \tag{2}$$

By our separation condition, there is a function G in A such that

$$|G| \leq c/3 \quad \text{on } U \qquad \text{and} \qquad |G| \geq 3 \quad \text{on } X \backslash U. \tag{3}$$

Put $F = f + G(1-h)$. Clearly $F \in A$ and $F = f$ on S. For $x \in U$, by (3) we have

$$|F(x)| \geq |f(x)| - |G(x)|\,|1 - h(x)| \geq c/3 > 0.$$

For $x \in X \backslash U$, by (1), (2), and (3) we have

$$|F(x)| \geq |G(x)|\,|1 - h(x)| - |f(x)| \geq 1/2 > 0.$$

Thus $|F| > 0$ on X.

REMARK 1. Let A be a strongly logmodular algebra on a totally disconnected compact space X, that is, $\log |A^{-1}| = C_R(X)$, where A^{-1} denotes the set of invertible elements in A and $C_R(X)$ denotes the space of real continuous functions on X. Then A satisfies (#).

REMARK 2. For a general function algebra A, the assertion of Theorem 1 is not true. Put $X = \{(z,t); z$ is a complex number and t is a real number with $|z| = 1 - |t|$ and $|t| \leq 1\}$. Let $A = \{g \in C(X)$; for each t with $|t| < 1$, $g(z,t)$ has a continuous holomorphic extension to $\{|z| \leq 1 - |t|\}\}$. Then $S = \{(z,0); |z| = 1\}$ is a peak set for A. Put $f(z,t) = (1 - |t|)z$. Then $f \in A$ and $|f| > 0$ on S, but it is easy to see that there is no $F \in A$ with $F = f$ on S and $|F| > 0$ on X.

For a weak peak set S, put $A_S = \{f \in C(X); f|S \in A|S\}$. Then A_S is a closed subalgebra of $C(X)$. For $\psi \in C(X)$ and $B \subset C(X)$, put $d(\psi, B) = \inf\{\|\psi - f\|; f \in B\}$. For $\psi \in C(X)$ and $F \subset X$, put $\|\psi\|_F = \sup\{|\psi(x)|; x \in F\}$. The following theorem is a generalization of [**8**, Theorem 3.1].

THEOREM 2. *Suppose that A satisfies separation condition (#). Let S be a weak peak set for A, and let u be a function in $C(X)$ such that $\|u\| = 1$, $|u| = 1$ on S, and $d(u, A_S) < 1$. Then there is a function $\tilde{u}$ in $C(X)$ such that $|\tilde{u}| = |u|$ on X, $\tilde{u} = u$ on S, and $d(\tilde{u}, A) < 1$.*

PROOF. Since $d(u, A_S) < 1$, there exists a function f in A such that $\|u - f\|_S < 1$. Since $|u| = 1$ on S, $|f| > 0$ on S. By Theorem 1, there is a function F in A such that $F = f$ on S, and

$$|F| > 0 \quad \text{on } X. \tag{1}$$

Since $\|u - F\|_S = \|u - f\|_S < 1$, there is an open and closed subset U of X such that $S \subset U$ and

$$\|u - F\|_U < 1. \tag{2}$$

Since S is a weak peak set, there is a peak set E for A such that $S \subset E \subset U$. Let $h \in A$ be a peaking function for E. Replacing h by high powers of $(1+h)/2$, we may assume that

$$|h| > 0 \quad \text{on } X \qquad \text{and} \qquad 0 < |hF| < 1 \quad \text{on } X \backslash U. \tag{3}$$

Put

$$\tilde{u} = \begin{cases} uh/|h| & \text{on } U, \\ |u|hF/|hF| & \text{on } X\backslash U. \end{cases}$$

By (3), $\tilde{u} \in C(X)$. Clearly $|\tilde{u}| = |u|$ on X and $\tilde{u} = u$ on S. To prove $d(\tilde{u}, A) < 1$, it is sufficient to prove $\|\tilde{u} - hF\| < 1$, because $hF \in A$. We have

$$\begin{aligned} \|\tilde{u} - hF\|_{X\backslash U} &= \left\| \frac{|u|hF}{|hF|} - \frac{hF|hF|}{|hF|} \right\|_{X\backslash U} \\ &= \| \, |u| - |hF| \, \|_{X\backslash U} \\ &< 1 \quad \text{by (3) and } \|u\| \le 1. \end{aligned}$$

Also we have

$$\|\tilde{u} - hF\|_U = \left\| \frac{uh}{|h|} - \frac{hF|h|}{|h|} \right\|_U = \|u - F|h| \, \|_U.$$

Let $x \in U$. Then

$$\begin{aligned} |u(x) - F(x)|h(x)|| &= |(1 - |h(x)|)u(x) + |h(x)|(u(x) - F(x))| \\ &\le 1 - |h(x)| + |h(x)| \, \|u - F\|_U \quad \text{by } \|u\| \le 1, \|h\| \le 1 \\ &= 1 - |h(x)|\{1 - \|u - F\|_U\} \\ &< 1 \quad \text{by (2)}. \end{aligned}$$

Thus we get $\|\tilde{u} - hF\|_U < 1$, hence $\|\tilde{u} - hF\| < 1$.

If A is a strongly logmodular algebra, then A_S, where S is a weak peak set for A, is generated by A and $\{f^{-1}; f \in A \cap A_S^{-1}\}$ [**2**]. Younis used this property to prove his theorem [**8**, Theorem 3.1]. So the following theorem is another generalization of [**8**, Theorem 3.1].

THEOREM 3. *Suppose that A satisfies separation condition (#). Let S be a weak peak set for A such that A_S is generated by A and $\{f^{-1}; f \in A \cap A_S^{-1}\}$. If u is a function in $C(X)$ with $\|u\| \le 1$ and $d(u, A_S) < 1$, then there exists a function $\tilde{u}$ in $C(X)$ such that $|\tilde{u}| = |u|$ on X, $\tilde{u} = u$ on S, and $d(\tilde{u}, A) < 1$.*

NOTE. We do not assume $|u| = 1$ on S.

PROOF. By our assumption, there exist functions g in A and f in $A \cap A_S^{-1}$ such that

$$\|u - f^{-1}g\| < 1. \tag{1}$$

Since $f^{-1} \in A_S$, $|f^{-1}| > 0$ on X. By Theorem 1, there is a function F in A such that $F = f^{-1}$ on S and $|F| > 0$ on X. Put $\tilde{u} = ufF/|fF|$. Since $|fF| > 0$ on X, $\tilde{u} \in C(X)$. Clearly $|\tilde{u}| = |u|$ on X and $\tilde{u} = u$ on S. To prove $d(\tilde{u}, A) < 1$, suppose not. Since $\|\tilde{u}\| = \|u\| \le 1$, we have $d(\tilde{u}, A) = 1$. We note that the space of bounded linear functionals of $C(X)/A$ may be identified with $A^\perp$, the set of regular Borel measures μ on X such that $\int_X \phi \, d\mu = 0$ for every $\phi \in A$. Hence there exists $\mu \in A^\perp$ with the unit total variation, $\|\mu\| = 1$, such that

$$1 = \int_X \tilde{u} \, d\mu = \int_X \frac{ufF}{|fF|} \, d\mu.$$

Since $\|u\| \le 1$, we get $((ufF)/|fF|)\mu = |\mu|$, hence

$$ufF\mu = |fF| \, |\mu|. \tag{2}$$

Then

$$\begin{aligned} 1 &> \left| \int_X (u - f^{-1}g) \frac{fF}{\|fF\mu\|} \, d\mu \right| && \text{by (1)} \\ &= \frac{1}{\|fF\mu\|} \left| \int_X ufF \, d\mu \right| && \text{by } \mu \in A^{\perp} \\ &= \frac{1}{\|fF\mu\|} \int_X |fF| \, d|\mu| && \text{by (2)} \\ &= 1. \end{aligned}$$

This contradiction shows $d(\tilde{u}, A) < 1$.

3. Restricted left invertible Toeplitz operators. Let Ω be a bounded connected open subset of the plane whose boundary Γ consists of finitely many disjoint analytic Jordan curves. Identifying a function in $H^\infty(\Omega)$ with its boundary function, we may regard $H^\infty(\Omega)$ as an essentially supremum norm closed subalgebra of $L^\infty(m)$, where m is the arc length measure on Γ. A closed subspace $\mathcal{M}$ of $L^2(m)$ with $\mathcal{M} \neq \{0\}$ is called invariant if $H^\infty(\Omega)\mathcal{M} \subset \mathcal{M}$, and it is called reducing if $H^\infty(\Omega)\mathcal{M} \subset \mathcal{M}$ and $\overline{H^\infty(\Omega)}\mathcal{M} \subset \mathcal{M}$. An invariant subspace of $L^2(m)$ is called simple if it contains no reducing subspaces. For a closed subspace $\mathcal{M}$, $P_{\mathcal{M}}$ denotes the orthogonal projection of $L^2(m)$ onto $\mathcal{M}$. For a function ψ in $L^\infty(m)$, put $T_\psi^{\mathcal{M}}(f) = P_{\mathcal{M}}(\psi f)$ for every f in $\mathcal{M}$. $\mathcal{T}_\psi = \{T_\psi^{\mathcal{M}}; \mathcal{M} \text{ is a simply invariant subspace}\}$ is called a family of Toeplitz operators with a symbol ψ. In the case that Ω is the unit open disk, by Beurling's theorem, $T_\psi^{\mathcal{M}}$ is unitarily equivalent to the usual Toeplitz operator T_ψ. We call $\mathcal{T}_\psi$ left invertible if $T_\psi^{\mathcal{M}}$ is a left invertible operator on $\mathcal{M}$ for every simply invariant subspace $\mathcal{M}$. In [**1**, Theorem 4.1], Abrahamse proved the following theorem which is a generalization of the Devinatz-Rabindranathan theorem (see [**6**, p. 119]).

THEOREM 4. *Let ψ be a function in $L^\infty(m)$ with $|\psi| = 1$ a.e. dm. Then the following conditions are equivalent:*

(i) *$\mathcal{T}_\psi$ is left invertible.*

(ii) $d(\psi, H^\infty(\Omega)) < 1$.

We note that by his proof, (i) $\Rightarrow$ (ii) is true for every $\psi \in L^\infty(m)$ with $\|\psi\| \leq 1$.

Let X be the maximal ideal space of $L^\infty(m)$. Then X is a totally disconnected compact Hausdorff space [**4**, p. 190]. Identifying a function in $L^\infty(m)$ with its Gelfand transform, we have $L^\infty(m) = C(X)$. We may consider $H^\infty(\Omega)$ as a function algebra on X [**4**, p. 123]. Let S be a weak peak subset of X for $H^\infty(\Omega)$. A family of Toeplitz operators $\mathcal{T}_\psi$, $\psi \in C(X)$, is called S-restricted left invertible if there is a function Ψ in $C(X)$ such that $\Psi = \psi$ on S and $\mathcal{T}_\Psi$ is left invertible. In the case that Ω is the open unit disk, Clancey-Gosselin-Younis [**3**, **7**] gave a characterization of S-restricted left invertible Toeplitz operators as follows: If $|\psi| = 1$ a.e. dm, then the Toeplitz operator T_ψ is S-restricted left invertible if and only if $d(\psi, H_S^\infty) < 1$. We shall give a generalization of the above theorem to multiply connected domains as an application of §2.

THEOREM 5. *Let S be a weak peak subset of X for $H^\infty(\Omega)$. Let ψ be a function in $C(X)$ with $|\psi| = 1$ on S. Then the following conditions are equivalent:*
(i) *$\mathcal{T}_\psi$ is S-restricted left invertible.*
(ii) $d(\psi, H^\infty(\Omega)_S) < 1$.

PROOF. (ii) $\Rightarrow$ (i) Since X is totally disconnected, we may assume that $|\psi| = 1$ on X. We note that $A = H^\infty(\Omega)$ satisfies separation condition (#) in §2 [**5**, p. 119]. Then $H^\infty(\Omega)$, ψ, and S satisfy all assumption of Theorem 2. Hence there exists a function Ψ in $C(X)$ such that $|\Psi| = 1$ on X, $\Psi = \psi$ on S, and $d(\Psi, H^\infty(\Omega)) < 1$. By Theorem 4, $\mathcal{T}_\Psi$ is left invertible. So (i) holds.

(i) $\Rightarrow$ (ii) Suppose that $\mathcal{T}_\psi$ is S-restricted left invertible. By our definition, there is a function Ψ in $C(X)$ such that $\Psi = \psi$ on S and $\mathcal{T}_\Psi$ is left invertible. First we shall show that there is a function h in $H^\infty(\Omega)$ such that

$$h = 1 \text{ on } S, \quad \|\Psi h\| \le 1, \quad \text{and} \quad |h| > 0 \text{ on } X. \tag{1}$$

To prove this, put

$$\phi = \max\{|\Psi|, 1\}. \tag{2}$$

Then $\phi \in C(X)$, $\phi \ge 1$ on X, and $\phi = 1$ on S. By [**5**, p. 58], there is a function g in $H^\infty(\Omega)$ such that $g = 1$ on S and

$$|g| \le 1/\phi \text{ on } X. \tag{3}$$

Then $\|g\| = 1$. Take a positive integer N with

$$N \ge 4 \quad \text{and} \quad \|\Psi\|(5/6)^N \le 1. \tag{4}$$

Put $h = ((g+2)/3)^N$. Then $h \in H^\infty(\Omega)$, $h = 1$ on S, and $|h| > 0$ on X. If $x \in X$ with $|g(x)| \le 1/2$, then

$$|\Psi(x)h(x)| \le \|\Psi\|(5/6)^N \le 1 \quad \text{by (4)}.$$

If $x \in X$ with $1/2 \le |g(x)| \le 1$, then

$$\begin{aligned} |\Psi(x)h(x)| &\le ((|g(x)|+2)/3)^N |\phi(x)| && \text{by (2)} \\ &\le ((|g(x)|+2)/3)^N / |g(x)| && \text{by (3)} \\ &\le 1. \end{aligned}$$

The last inequality follows from $((t+2)/3)^N \le t$ for $1/2 \le t \le 1$ and $N \ge 4$. Thus $\|\Psi h\| \le 1$ and we get (1).

We put $\Phi = \Psi h \in L^\infty(m)$. By (1),

$$\|\Phi\| \le 1 \quad \text{and} \quad \Phi = \psi \text{ on } S. \tag{5}$$

Also we have that $\mathcal{T}_\Phi$ is left invertible. To see this, let $\mathcal{M}$ be a simply invariant subspace of $L^2(m)$. Since $h \in H^\infty(\Omega)$ and $|h| > 0$ on X, $T_h^{\mathcal{M}}$ is left invertible. Since $\mathcal{T}_\Psi$ is left invertible, $T_\Phi^{\mathcal{M}} = T_\Psi^{\mathcal{M}} T_h^{\mathcal{M}}$ is left invertible, so $\mathcal{T}_\Phi$ is left invertible. As noted after Theorem 4, we get $d(\Phi, H^\infty(\Omega)) < \|\Phi\| \le 1$. Hence

$$\begin{aligned} d(\psi, H^\infty(\Omega)_S) &= d(\Phi, H^\infty(\Omega)_S) && \text{by (5)} \\ &\le d(\Phi, H^\infty(\Omega)) < 1. \end{aligned}$$

This completes the proof.

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