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SOME FUNCTIONAL EQUATIONS IN BANACH ALGEBRAS AND AN APPLICATION

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ABSTRACT. In this paper some results concerning certain functional equations in complex Banach algebras are presented. One of these results is used to prove an abstract generalization of the classical Jordan-Neumann characterization of pre-Hilbert space.

This paper is a continuation of our earlier work [**6**, **7**]. Throughout this paper all Banach algebras and vector spaces are over the complex field **C**. Our terminology and notation will be the same as in [**7**]. For Banach algebras and Banach $*$-algebras we refer to [**1** and **4**]. Our first few results characterize some additive functions.

THEOREM 1. *Let A be a Banach $*$-algebra with identity e. Let λ and μ be automorphisms or antiautomorphisms (i.e. $\lambda(ab) = \lambda(b)\lambda(a)$) of A (any combination is allowed). If $f\colon A \to A$ is an additive function such that $f(a) = \lambda(a)f(a^{-1})\mu(a)$ for all normal invertible elements a of A, then $2f(b) = \lambda(b)f(e) + f(e)\mu(b)$ for all $b \in A$.*

PROOF. Let us first assume that for the function f

$$f(e) = 0 \tag{1}$$

holds and let us prove that in this case

$$f(a) = 0 \tag{2}$$

is fulfilled for all $a \in A$. Since f is by the assumption additive, (2) will be proved if we prove that (2) holds for all normal elements. Therefore let $a \in A$ be an arbitrary normal element. One can choose rational numbers p and q such that b^{-1} and $(e-b)^{-1}$ exist, where $b = pe + qa$. Hence $f(a) = 0$ will be proved if we prove that $f(b) = 0$. Now according to the requirements of the theorem and (1) we have

$$\begin{aligned} f(b) &= \lambda(b)f(b^{-1})\mu(b) = \lambda(b)f(b^{-1}(e-b))\mu(b) \\ &= \lambda(b)\lambda(b^{-1}(e-b))f(e-b)^{-1}b)\mu(b^{-1}(e-b))\mu(b) \\ &= \lambda(e-b)f((e-b)^{-1} - e)\mu(e-b) \\ &= \lambda(e-b)\lambda((e-b)^{-1})f(e-b)\mu((e-b)^{-1})\mu(e-b) \\ &= -f(b). \end{aligned}$$

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Hence $f(b) = -f(b)$ which implies $f(a) = 0$ for an arbitrary normal element a and so (2) is proved. Let us now prove the theorem in full generality. For this purpose we introduce a function $g\colon A \to A$ defined by

$$g(a) = f(a) - \tfrac{1}{2}(\lambda(a)f(e) + f(e)\mu(a)),$$

where f, λ, and μ are such that the requirements of the theorem are fulfilled. It is obvious that g is additive. A simple calculation shows that $g(a) = \lambda(a)g(a^{-1})\mu(a)$ holds for all normal invertible elements $a \in A$. Since $g(e) = 0$ we have $g(a) = 0$ for all $a \in A$. In other words

$$f(a) = \tfrac{1}{2}(\lambda(a)f(e) + f(e)\mu(a)).$$

The proof of the theorem is complete.

Using a similar approach as in the proof of Theorem 1 one can prove the following result.

THEOREM 2. *Let A be a Banach $*$-algebra with identity e. Let λ and μ be automorphisms or antiautomorphisms of A (any combination is allowed) such that $\lambda(a)\mu(a) = \mu(a)\lambda(a)$ for all normal invertible elements a of A. If $f\colon A \to A$ is an additive function such that $f(a) = \lambda(a)\mu(a)f(a^{-1})$ for all normal invertible elements a of A, then $2f(b) = (\lambda(b) + \mu(b))f(e)$ for all $b \in A$.*

One can now state two theorems in the spirit of Theorem 1 and Theorem 2 for arbitrary Banach algebras. More precisely, using a similar approach as in the proof of Theorem 1 one can prove the following two results.

THEOREM 3. *Let A be a Banach algebra with identity e. Let λ and μ be automorphisms or antiautomorphisms (any combination is allowed). If $f\colon A \to A$ is an additive function such that $f(a) = \lambda(a)f(a^{-1})\mu(a)$ for all invertible elements a of A, then $2f(b) = \lambda(b)f(e) + f(e)\mu(b)$ holds for all $b \in A$.*

THEOREM 4. *Let A be a Banach algebra with identity e. Let λ and μ be automorphisms or antiautomorphisms (any combination is allowed) such that $\lambda(a)\mu(a) = \mu(a)\lambda(a)$ for all invertible elements $a \in A$. If $f\colon A \to A$ is an additive function such that $f(a) = \lambda(a)\mu(a)f(a^{-1})$ for all invertible elements a of A, then $2f(b) = (\lambda(b) + \mu(b))f(e)$ holds for all $b \in A$.*

The following corollaries are immediate consequences of the theorems above (by appropriately choosing λ and μ).

COROLLARY 1. *Let A be a Banach $*$-algebra with identity e and let $f\colon A \to A$ be an additive function. The following statements are fulfilled.*

1° *If $f(a) = af(a^{-1})a^*$ holds for all normal invertible elements a of A, then f is of the form $2f(b) = bf(e) + f(e)b^*$.*

2° *If $f(a) = af(a^{-1})a$ holds for all normal invertible elements a of A, then f is of the form $2f(b) = bf(e) + f(e)b$.*

3° *If $f(a) = a^2f(a^{-1})$ holds for all normal invertible elements a of A, then f is of the form $f(b) = bf(e)$.*

4° *If $f(a) = a^*af(a^{-1})$ holds for all normal invertible elements a of A, then f is of the form $f(b) = hf(e)$, where $b = h + ik$ and h and k are hermitian.*

COROLLARY 2. *Let A be a Banach algebra with identity e and let $f\colon A \to A$ be an additive function. The following statements are fulfilled.*

1° *If* $f(a) = af(a^{-1})a$ *holds for all invertible elements* a *of* A, *then* f *is of the form* $2f(b) = bf(e) + f(e)b$.

2° *If* $f(a) = a^2 f(a^{-1})$ *holds for all invertible elements* a *of* A, *then* f *is of the form* $f(b) = bf(e)$.

It should be mentioned that the results included in the corollaries above have been proved in our earlier paper **[7]** under a somewhat more complicated approach and a much stronger assumption that A is a hermitian Banach $*$-algebra (that is each hermitian element has real spectrum).

In the continuation we present the following results.

THEOREM 5. *Let* A *be a Banach* $*$*-algebra with identity* e *and let* $f\colon A \to A$ *be an additive function. Then the following statements are fulfilled.*

1° *If* $f(a) = -af(a^{-1})a^*$ *holds for all normal invertible elements* a *of* A, *then* f *is of the form* $2if(b) = bf(ie) - f(ie)b^*$.

2° *If* $f(a) = -a^*af(a^{-1})$ *hodlds for all normal invertible elements* a *of* A, *then* f *is of the form* $f(b) = kf(ie)$, *where* $b = h + ik$ *and* h *and* k *are hermitian.*

PROOF. If we introduce a function $g\colon A \to A$ by the relation $g(a) = f(ia)$, then 1° follows from statement 1° of Corollary 1 and 2° from statement 4° of Corollary 1.

We conclude our discussion of additive functions with the following results.

THEOREM 6. *Let* A *be a Banach* $*$*-algebra with identity* e *and let* $f\colon A \to A$, $g\colon A \to A$ *be additive functions. Then the following statements are fulfilled.*

1° *If* $f(a) = ag(a^{-1})a^*$ *holds for all normal invertible elements* a *of* A, *then* f *and* g *are of the form*

$$2f(b) = b(f(e) - if(ie)) + (f(e) + if(ie))b^*,$$
$$2g(b) = b(f(e) + if(ie)) + (f(e) - if(ie))b^*.$$

2° *If* $f(a) = a^*ag(a^{-1})$ *holds for all normal invertible elements* a *of* A, *then* f *and* g *are of the form*

$$f(b) = hf(e) + kf(ie), \qquad g(b) = hf(e) - kf(ie),$$

where $b = h + ik$ *and* h *and* k *are hermitian.*

PROOF. Let us prove 1°. From

$$f(a) = ag(a^{-1})a^* \tag{3}$$

we obtain that also

$$g(a) = af(a^{-1})a^* \tag{4}$$

for all normal invertible $a \in A$. Let us introduce F and G by $F(a) = f(a) + g(a)$, $G(a) = f(a) - g(a)$. F and G are obviously additive and from (3) and (4) one obtains easily that $F(a) = aF(a^{-1})a^*$ and that $G(a) = -aG(a^{-1})a^*$ holds for all normal invertible $a \in A$. Hence according to statement 1° of Corollary 1 and statement 1° of Theorem 5 we have

$$2F(a) = aF(e) + F(e)a^*, \qquad 2iG(a) = aG(ie) - G(ie)a^*$$

for all $a \in A$. Since $g(e) = f(e)$ and $g(ie) = -f(ie)$, it follows that $f(a) + g(a) = af(e) + f(e)a^*$ and $f(a) - g(a) = -aif(ie) + if(ie)a^*$ for all $a \in A$. This proves

1°. Similarly one can prove that 2° follows from statement 4° of Corollary 1 and statement 2° of Theorem 5.

This completes our discussion of additive functions. One may also use statement 1° of Corollary 1 to generalize the well-known Jordan-Neumann characterization of pre-Hilbert space. The rest of our paper does this.

THEOREM 7. *Let A be a Banach $*$-algebra with identity e and let X be a vector space which is also a unitary left A-module. Suppose there exists a mapping $Q\colon X \to A$ with the properties*

(i) $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$ *for all pairs* $x,y\in X$,

(ii) $Q(ax)=aQ(x)a^*$ *for all* $x\in X$ *and all normal invertible* a *of* A.

Under these conditions for the mapping $B(\cdot,\cdot)\colon X\times X\to A$ defined by the relation

$$B(x,y)=\tfrac{1}{4}(Q(x+y)-Q(x-y))+\tfrac{i}{4}(Q(x+iy)-Q(x-iy))$$

the following statements are fulfilled.

1° $B(\cdot,\cdot)$ *is additive in both arguments.*

2° $B(ax,y)=aB(x,y)$, $B(x,ay)=B(x,y)a^*$ *for all pairs* $x,y\in X$ *and all* $a\in A$.

3° $Q(x)=B(x,x)$ *for all* $x\in X$.

The result above was proved in [**7**] under the stronger assumption that A is a hermitian Banach $*$-algebra (see also [**6**]). If A is the complex number field, then Theorem 7 reduces to S. Kurepa's extension of Jordan-Neumann theorem (see [**2**, **3**] and also [**5**]).

The proof of Theorem 7 we shall omit since it goes through in the same way as in [**7**] with the only exception that for the proof of statement 2° one has to use statement 1° of Corollary 1 instead of Theorem 1.3 in [**7**].

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