SOME FUNCTIONAL EQUATIONS IN BANACH ALGEBRAS
AND AN APPLICATION

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ABSTRACT. In this paper some results concerning certain functional equa-
tions in complex Banach algebras are presented. One of these results is used
to prove an abstract generalization of the classical Jordan-Neumann charac-
terization of pre-Hilbert space.

This paper is a continuation of our earlier work [6, 7]. Throughout this paper all
Banach algebras and vector spaces are over the complex field C. Our terminology
and notation will be the same as in [7]. For Banach algebras and Banach *-algebras
we refer to [1 and 4]. Our first few results characterize some additive functions.

THEOREM 1. Let A be a Banach *-algebra with identity e. Let λ and μ be au-
tomorphisms or antiautomorphisms (i.e. λ(ab) = λ(b)λ(a)) of A (any combination
is allowed). If f : A → A is an additive function such that f(a) = λ(a)f(a−1)μ(a)
for all normal invertible elements a of A, then 2f(b) = λ(b)f(e) + f(e)μ(b) for all
b ∈ A.

PROOF. Let us first assume that for the function f

(1) f(e) = 0

holds and let us prove that in this case

(2) f(a) = 0

is fulfilled for all a ∈ A. Since f is by the assumption additive, (2) will be proved
if we prove that (2) holds for all normal elements. Therefore let a ∈ A be an
arbitrary normal element. One can choose rational numbers p and q such that b−1
and (e − b)−1 exist, where b = pe + qa. Hence f(a) = 0 will be proved if we prove
that f(b) = 0. Now according to the requirements of the theorem and (1) we have

\[ f(b) = \lambda(b)f(b^{-1})\mu(b) = \lambda(b)f(b^{-1}(e - b))\mu(b) \]
\[ = \lambda(b)\lambda(b^{-1}(e - b))f(e - b)^{-1}b\mu(b^{-1}(e - b))\mu(b) \]
\[ = \lambda(e - b)f((e - b)^{-1} - e)\mu(e - b) \]
\[ = \lambda(e - b)\lambda((e - b)^{-1})f(e - b)\mu((e - b)^{-1})\mu(e - b) \]
\[ = -f(b). \]
Hence \( f(b) = -f(b) \) which implies \( f(a) = 0 \) for an arbitrary normal element \( a \) and so (2) is proved. Let us now prove the theorem in full generality. For this purpose we introduce a function \( g: A \to A \) defined by
\[
g(a) = f(a) - \frac{1}{2} \left( \lambda(a)f(e) + f(e)\mu(a) \right),
\]
where \( f, \lambda, \) and \( \mu \) are such that the requirements of the theorem are fulfilled. It is obvious that \( g \) is additive. A simple calculation shows that \( g(a) = \lambda(a)g(a^{-1})\mu(a) \) holds for all normal invertible elements \( a \in A \). Since \( g(e) = 0 \) we have \( g(a) = 0 \) for all \( a \in A \). In other words
\[
f(a) = \frac{1}{2} \left( \lambda(a)f(e) + f(e)\mu(a) \right).
\]
The proof of the theorem is complete.

Using a similar approach as in the proof of Theorem 1 one can prove the following result.

**Theorem 2.** Let \( A \) be a Banach \(*\)-algebra with identity \( e \). Let \( \lambda \) and \( \mu \) be automorphisms or antiautomorphisms of \( A \) (any combination is allowed) such that \( \lambda(a)\mu(a) = \mu(a)\lambda(a) \) for all normal invertible elements \( a \) of \( A \). If \( f: A \to A \) is an additive function such that \( f(a) = \lambda(a)\mu(a)f(a^{-1}) \) for all normal invertible elements \( a \) of \( A \), then \( 2f(b) = (\lambda(b) + \mu(b))f(e) \) for all \( b \in A \).

One can now state two theorems in the spirit of Theorem 1 and Theorem 2 for arbitrary Banach algebras. More precisely, using a similar approach as in the proof of Theorem 1 one can prove the following two results.

**Theorem 3.** Let \( A \) be a Banach algebra with identity \( e \). Let \( \lambda \) and \( \mu \) be automorphisms or antiautomorphisms (any combination is allowed). If \( f: A \to A \) is an additive function such that \( f(a) = \lambda(a)f(a^{-1})\mu(a) \) for all invertible elements \( a \) of \( A \), then \( 2f(b) = \lambda(b)f(e) + f(e)\mu(b) \) holds for all \( b \in A \).

**Theorem 4.** Let \( A \) be a Banach algebra with identity \( e \). Let \( \lambda \) and \( \mu \) be automorphisms or antiautomorphisms (any combination is allowed) such that \( \lambda(a)\mu(a) = \mu(a)\lambda(a) \) for all invertible elements \( a \in A \). If \( f: A \to A \) is an additive function such that \( f(a) = \lambda(a)\mu(a)f(a^{-1}) \) for all invertible elements \( a \) of \( A \), then \( 2f(b) = (\lambda(b) + \mu(b))f(e) \) holds for all \( b \in A \).

The following corollaries are immediate consequences of the theorems above (by appropriately choosing \( \lambda \) and \( \mu \)).

**Corollary 1.** Let \( A \) be a Banach \(*\)-algebra with identity \( e \) and let \( f: A \to A \) be an additive function. The following statements are fulfilled.

1° If \( f(a) = af(a^{-1})a^* \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( 2f(b) = bf(e) + f(e)b^* \).

2° If \( f(a) = af(a^{-1})a \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( 2f(b) = bf(e) + f(e)b \).

3° If \( f(a) = a^2f(a^{-1}) \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( f(b) = bf(e) \).

4° If \( f(a) = a^*af(a^{-1}) \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( f(b) = hf(e) \), where \( b = h + ik \) and \( h \) and \( k \) are hermitian.

**Corollary 2.** Let \( A \) be a Banach algebra with identity \( e \) and let \( f: A \to A \) be an additive function. The following statements are fulfilled.
1° If \( f(a) = af(a^{-1})a \) holds for all invertible elements \( a \) of \( A \), then \( f \) is of the form \( 2f(b) = bf(e) + f(e)b \).

2° If \( f(a) = a^2f(a^{-1}) \) holds for all invertible elements \( a \) of \( A \), then \( f \) is of the form \( f(b) = bf(e) \).

It should be mentioned that the results included in the corollaries above have been proved in our earlier paper [7] under a somewhat more complicated approach and a much stronger assumption that \( A \) is a hermitian Banach \(*\)-algebra (that is each hermitian element has real spectrum).

In the continuation we present the following results.

**THEOREM 5.** Let \( A \) be a Banach \(*\)-algebra with identity \( e \) and let \( f: A \rightarrow A \) be an additive function. Then the following statements are fulfilled.

1° If \( f(a) = -af(a^{-1})a^* \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( 2if(b) = b(f(e) - if(ie)) + (f(e) + if(ie))b^* \).

2° If \( f(a) = -a^*af(a^{-1}) \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) is of the form \( f(b) = kf(ie) \), where \( b = h + ik \) and \( h \) and \( k \) are hermitian.

**PROOF.** If we introduce a function \( g: A \rightarrow A \) by the relation \( g(a) = f(ia) \), then 1° follows from statement 1° of Corollary 1 and 2° from statement 4° of Corollary 1.

We conclude our discussion of additive functions with the following results.

**THEOREM 6.** Let \( A \) be a Banach \(*\)-algebra with identity \( e \) and let \( f: A \rightarrow A \), \( g: A \rightarrow A \) be additive functions. Then the following statements are fulfilled.

1° If \( f(a) = ag(a^{-1})a^* \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) and \( g \) are of the form

\[
2f(b) = b(f(e) - if(ie)) + (f(e) + if(ie))b^*,
\]

\[
2g(b) = b(f(e) + if(ie)) + (f(e) - if(ie))b^*.
\]

2° If \( f(a) = a^*ag(a^{-1}) \) holds for all normal invertible elements \( a \) of \( A \), then \( f \) and \( g \) are of the form

\[
f(b) = hf(e) + kf(ie),
\]

\[
g(b) = hf(e) - kf(ie),
\]

where \( b = h + ik \) and \( h \) and \( k \) are hermitian.

**PROOF.** Let us prove 1°. From

\[
f(a) = ag(a^{-1})a^*
\]

we obtain that also

\[
g(a) = af(a^{-1})a^*
\]

for all normal invertible \( a \in A \). Let us introduce \( F \) and \( G \) by \( F(a) = f(a) + g(a), \)

\( G(a) = f(a) - g(a) \). \( F \) and \( G \) are obviously additive and from (3) and (4) one obtains easily that \( F(a) = aF(a^{-1})a^* \) and that \( G(a) = -aG(a^{-1})a^* \) holds for all normal invertible \( a \in A \). Hence according to statement 1° of Corollary 1 and statement 1° of Theorem 5 we have

\[
2F(a) = aF(e) + F(e)a^*,
\]

\[
2iG(a) = aG(ie) - G(ie)a^*
\]

for all \( a \in A \). Since \( g(e) = f(e) \) and \( g(ie) = -f(ie) \), it follows that \( f(a) + g(a) = af(e) + f(e)a^* \) and \( f(a) - g(a) = -af(ie) + if(ie)a^* \) for all \( a \in A \). This proves
1°. Similarly one can prove that 2° follows from statement 4° of Corollary 1 and statement 2° of Theorem 5.

This completes our discussion of additive functions. One may also use statement 1° of Corollary 1 to generalize the well-known Jordan-Neumann characterization of pre-Hilbert space. The rest of our paper does this.

**Theorem 7.** Let A be a Banach *-algebra with identity e and let X be a vector space which is also a unitary left A-module. Suppose there exists a mapping Q: X → A with the properties

(i) Q(x + y) + Q(x − y) = 2Q(x) + 2Q(y) for all pairs x, y ∈ X,
(ii) Q(ax) = aQ(x)a* for all x ∈ X and all normal invertible a of A.

Under these conditions for the mapping B(-, ·): X × X → A defined by the relation

\[ B(x, y) = \frac{1}{4}(Q(x + y) - Q(x - y)) + \frac{i}{4}(Q(x + iy) - Q(x - iy)) \]

the following statements are fulfilled.

1° B(-, ·) is additive in both arguments.

2° B(ax, x) = aB(x, x), B(x, ay) = B(x, y)a* for all pairs x, y ∈ X and all a ∈ A.

3° Q(x) = B(x, x) for all x ∈ X.

The result above was proved in [7] under the stronger assumption that A is a hermitian Banach *-algebra (see also [6]). If A is the complex number field, then Theorem 7 reduces to S. Kurepa’s extension of Jordan-Neumann theorem (see [2, 3] and also [5]).

The proof of Theorem 7 we shall omit since it goes through in the same way as in [7] with the only exception that for the proof of statement 2° one has to use statement 1° of Corollary 1 instead of Theorem 1.3 in [7].

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REFERENCES