SIMILARITY OF A LINEAR STRICT SET-CONTRACTION
AND THE RADIUS OF THE ESSENTIAL SPECTRUM

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ABSTRACT. If $A$ is a bounded linear operator on a Hilbert space, define $r_*(A)$, the essential spectral radius of $A$, by

$$r_*(A) := \sup\{|\lambda|: \lambda \in \text{ess}(A) = \text{essential spectrum of } A\}. $$

It is shown that

$$r_*(A) = \inf\{a(S^{-1}AS)|S: H \rightarrow H \text{ is a bounded invertible linear map}\},$$

where $a$ is the Kuratowski measure of noncompactness. As a consequence, a characterization of the similarity of a linear strict set-contraction is obtained.

1. This paper applies a classical result of Rota with known results about Browder’s essential spectrum to give a new formula for the radius of the essential spectrum of a bounded linear operator in Hilbert space.

Throughout the paper, $H$ will denote a complex Hilbert space with norm $|| \cdot ||$. By an operator we always mean a bounded linear transformation on $H$. The identity operator is denoted by $I$. The spectrum of an operator $A$ is denoted by $\sigma(A)$ and the spectral radius is denoted by $r(A)$. Recall that an operator $A$ is called a strict contraction if $||A|| < 1$ [2, p. 82]. The classical result of Rota’s similarity theorem [6; 2, p. 81] asserts that an operator on $H$ is similar to a strict contraction if and only if its spectrum is included in the interior of the unit disc. There is an elegant quantitative version of Rota’s theorem [2, p. 77]; it asserts that the spectral radius of $A$ is always equal to the infimum of norms of all conjugates (i.e., transformation by similarities) of $A$. Seeking to characterize the similarity of a linear strict set-contraction (the perturbation of a strict contraction by a compact operator is a strict set-contraction), we may therefore raise the question: What kind of spectrum included in the interior of unit disc has to be similar to a linear strict set-contraction? Thanks to the works of Kuratowski [3], Browder [1], Nussbaum [5], and Leggett [4], we are capable of solving this full problem. Our main results are proved in §3 and §2 contains some preliminary notions and lemmas.

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2. We first recall the “measure of noncompactness,” a notion which was introduced by Kuratowski in 1930 [3].

Let $\Omega$ be a nonempty subset of $H$. Kuratowski [3] defined the measure of noncompactness of $\Omega$, in symbols $a(\Omega)$, to be

$$\inf\{\varepsilon > 0: \Omega \text{ can be covered by a finite number of sets of diameter less than or equal to } \varepsilon\}. $$

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It follows immediately that a subset \( \Omega \) of \( H \) has a compact closure if and only if \( \alpha(\Omega) = 0 \). Closely associated with the notion of the measure of noncompactness is the concept of "k-set-contraction," defined as follows. Let \( T \) be a continuous map from \( \Omega \) into \( H \); then \( T \) is called a k-set-contraction if there exists a constant \( k \geq 0 \) such that for any nonempty bounded set \( D \subset H \) we have \( \alpha(T(D)) \leq k \alpha(D) \). If \( T \) is a k-set-contraction, Nussbaum [5] defined the measure of noncompactness of \( T \), in symbols \( \alpha(T) \), to be

\[
\inf\{k > 0: T \text{ is a } k\text{-set-contraction}\}.
\]

If \( \alpha(T) < 1 \), then \( T \) is called a strict set-contraction. If \( A := C + S \), where \( C \) is a compact operator and \( S \) a strict contraction, then \( A \) is a strict set-contraction.

We may recall another notion introduced by Browder [1], that of the essential spectrum of an operator. Browder defined the essential spectrum of an operator \( A \), in symbols \( \text{ess}(A) \), to be the set of \( \lambda \in \sigma(A) \) such that at least one of the following conditions holds:

(i) \( R(\lambda I - A) \), the range of \( \lambda I - A \), is not closed.
(ii) \( \lambda \) is a limit point of \( \sigma(A) \).
(iii) \( \bigcup_{n=1}^{\infty} N(\lambda I - A)^n \) is infinite dimensional, where \( N(\lambda I - A)^n \) denotes the null space of \( (\lambda I - A)^n \).

If \( A \) is a an operator on \( H \), define \( r_e(A) \), the essential spectral radius of \( A \), by

\[
r_e(A) := \sup\{|\lambda|: \lambda \in \text{ess}(A)\}.
\]

Nussbaum proved in [5] that

\[
r_e(A) = \lim_{n \to \infty} \left( \alpha(A^n) \right)^{1/n}.
\]

Note that the above formula should be extended to read

\[
r_e(A) = \inf_{n \geq 1} \left( \alpha(A^n) \right)^{1/n} \leq \alpha(A).
\]

For an operator \( A \), we observe that \( A \) is a \( \|A\| \)-set-contraction and hence \( \alpha(A) \leq \|A\| \).

The following lemmas will be needed in the proofs of our results. When we say an operator is finite dimensional, we shall mean its range is finite dimensional.

**Lemma 1 (Nussbaum [5]).** Let \( A \) be an operator on \( H \) and \( r > r_e(A) \). Then there exists a finite dimensional operator \( F \) on \( H \) such that \( \sigma(A + F) \subset \{\lambda \in C: |\lambda| \leq r\} \).

**Lemma 2.** Similar operators on \( H \) have the same essential spectrum.

**Proof.** Let \( A \) be an operator on \( H \) and \( B = P^{-1}AP \) for some invertible operator \( P \) on \( H \). Then the conclusion follows from the following identities. (The identity (ii) below is a known result.)

(i) \( P^{-1}[R(\lambda I - A)] = R(\lambda I - B) \), \( P^{-1}[\text{cl}(\lambda I - A)] = \text{cl}(\lambda I - B) \), where \( \text{cl}(\lambda I - A) \) denotes the closure of \( (\lambda I - A)(H) \).
(ii) \( \sigma(A) = \sigma(B) \).
(iii) \( P^{-1}[N(\lambda I - A)^\nu] = N(\lambda I - B)^\nu \) for each positive integer \( \nu \).
3. Our main result is the following:

**Theorem 1.** Let $A$ be an operator on $H$. Then

$$r_e(A) = \inf \{ \alpha(S^{-1}AS) | S: H \to H \text{ is a bounded invertible linear map} \}. \tag{3.1}$$

**Proof.** Lemma 2 implies that if $S: H \to H$ is bounded and invertible,

$$r_e(A) = r_e(S^{-1}AS) \leq \alpha(S^{-1}AS). \tag{3.2}$$

Thus one has

$$r_e(A) \leq \inf \{ \alpha(S^{-1}AS) | S \text{ is one-one, onto, and linear} \}. \tag{3.3}$$

To prove the opposite inequality, take $\varepsilon > 0$ and use Lemma 1 to find a finite-dimensional operator $F$ such that

$$r(A + F) \leq r_e(A) + \varepsilon/2. \tag{3.4}$$

By Rota's theorem, there exists $S$ such that

$$\|S^{-1}(A + F)S\| \leq r(A + F) + \varepsilon/2 \leq r_e(A) + \varepsilon. \tag{3.5}$$

Finally we have (using basic properties of the seminorm $\alpha$)

$$\|S^{-1}(A + F)S\| \geq \alpha(S^{-1}(A + F)S) = \alpha(S^{-1}AS + S^{-1}FS) = \alpha(S^{-1}AS). \tag{3.6}$$

Here we have used that $\|B\| \geq \alpha(B)$ and $\alpha(B + C) = \alpha(B)$ for any compact linear map $C$. This proves that

$$\alpha(S^{-1}AS) \leq r_e(A) + \varepsilon \tag{3.7}$$

and since $\varepsilon > 0$ was arbitrary, the proof is complete.

**Theorem 2.** Let $A$ be an operator on $H$. Then $A$ is similar to a linear strict set-contraction if and only if $r_e(A) < 1$.

**Proof.** This is immediate from Lemma 2 and Theorem 1 above.

**References**