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ORIENTATION PRESERVING ACTIONS OF FINITE ABELIAN GROUPS ON SPHERES

RONALD M. DOTZEL

ABSTRACT. If G is a finite Abelian group acting as a $\mathbf{Z}_{(\mathcal{P})}$-homology n-sphere X (where $\mathcal{P}$ is the set of primes dividing $|G|$), then there is an integer valued function $n(\ ,G)$ defined on the prime power subgroups H of G such that X^H has the $\mathbf{Z}_{(p)}$-homology of a sphere $S^{n(H,G)}$. We prove here that there exists a real representation R of G such that for any prime power subgroup H of G, $\dim(S(R^H)) = n(H,G)$ where $S(R^H)$ is the unit sphere of R^H, provided that $n - n(H,G)$ is even whenever H is a 2-subgroup of G.

0. Introduction. Suppose that G is a finite Abelian group and let $\mathcal{P}$ be the set of primes dividing $|G|$. If G acts on a finite CW-complex X which has the $\mathbf{Z}_{(\mathcal{P})}$-homology of S^n, then for any $p \in \mathcal{P}$ and p-subgroup H of G the fixed point set of H on X, X^H, has the $\mathbf{Z}_{(p)}$-homology of $S^{n(H,G)}$ for some integer $n(H,G) \geq -1$ (-1 signifies empty). This is a well-known consequence of Smith theory ([**2**, III or **1**, IV], e.g.). Thus we obtain in this way an integer valued function, $n(\ ,G)$, defined on the set of p-subgroups of G by $H \mapsto n(H,G)$ (note that $n(e,G) = n$). This function is called the "dimension function" and has a considerable literature (see [**5**, **3**, **6**, **7**, **9**]; [**10**] gives a related extensive bibliography).

The function $n(\ ,G)$ satisfies the following well-known conditions (see [**1**, XIII, 2.3; IV, 4.4, 4.7]):

1. (Borel Formula) If $H \leq K$ are both p-subgroups of G and $K/H = \mathbf{Z}_p + \mathbf{Z}_p$, then $n(H,G) - n(K,G) = \sum(n(K',G) - n(K,G))$ with the sum over all $H \leq K' \leq K$ such that $K'/H = \mathbf{Z}_p$.
2. If $H \leq K$ are p-subgroups of G, then $n(K,G) \leq n(H,G)$.
3. If $H \leq K$ are p-subgroups of G with $K/H = \mathbf{Z}_p$ and p odd, then $n(H,G) - n(K,G)$ is even.
4. If $H \leq K' \leq K$ are 2-subgroups of G such that $K/H = \mathbf{Z}_4$, $K'/H = \mathbf{Z}_2$, then $n(H,G) - n(K',G)$ is even.

For each $p \in \mathcal{P}$, let $G(p)$ denote the p-Sylow subgroup of G and set $N(\ ,G) = n(\ ,G) + 1$. The function $N(\ ,G)$ restricted to the subgroups of $G(p)$ will naturally be denoted by $N(\ ,G(p))$. In [**8**] it was shown that $N(\ ,G(p))$ is realized by a real representation $V(p)$ of $G(p)$ which means that for each $H \leq G(p)$, $\dim V(p)^H = N(H,G(p)) = N(H,G)$. If $S(V(p))$ denotes the unit sphere of $V(p)$, then $\dim S(V(p)^H) = n(H,G)$.

Here we are interested in the existence of a real representation R of the Abelian group G such that for any p-subgroup H of G (for any $p \in \mathcal{P}$), $\dim R^H = N(H,G)$. Thus R would be a simultaneous realization of the functions $N(\ ,G(p))$, $p \in \mathcal{P}$.

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It should be noted that in the special case where X^H is a homology sphere for *all* $H \leq G$, then $N(\ ,G)$ is defined on *all* subgroups of G and tom Dieck has shown in [**3**] that $N(\ ,G)$ is realized by a difference of representations. In general, this is best possible.

Clearly $N(\ ,G)$ satisfies conditions 1–4 if $n(\ ,G)$ does. We will denote $N(e,G)$ by N. As an example to show that some condition beyond 1–4 is needed, suppose $G = \mathbf{Z}_6$, $N = 2$, $N(\mathbf{Z}_3, G) = 0$, and $N(\mathbf{Z}_2, G) = 1$. Then $N(\ ,G)$ satisfies conditions 1–4 but there is no real representation of G which realizes these numbers simultaneously as dimensions. From now on we will assume the following "orientation preserving" condition holds, in addition to conditions 1–4:

5. If H is any 2-subgroup of G, then $N - N(H,G)$ is even.

We obtain the following theorem and corollary:

THEOREM. *Let G be a finite Abelian group and suppose $N(\ ,G)$ is a nonnegative integer valued function defined on the p-subgroups of G for all $p \mid |G|$, satisfying conditions* 1–5. *Then there exists a real representation R of G such that for any p-subgroup H of G, $p \mid |G|$, $\dim R^H = N(H,G)$. Furthermore, if $\overline{R}$ is another such representation of G then for all subgroups H of G,*

$$\dim R^H \equiv \dim \overline{R}^H \pmod 2.$$

COROLLARY. *Let G be a finite Abelian group and suppose the* 2*-Sylow subgroup of G is cyclic. If $N(\ ,G)$ is a nonnegative integer valued function defined on the p-subgroups of G, $p \mid |G|$ satisfying only conditions* 1–4, *then there exists a real representation R such that for any p-subgroup H of G, $p \mid |G|$, $\dim R^H = N(H,G)+1$.*

In §§1 and 2 we prove the theorem and corollary respectively. We thank the referee for several suggestions leading to an improved exposition.

1. Proof of the theorem. Let G be a finite Abelian group and suppose $N(\ ,G)$ is a nonnegative integer valued function on the p-subgroups of G, for all $p \in \mathcal{P}$, satisfying conditions 1–5. By [**8**], for each $p \in \mathcal{P}$ there is a representation $V(p)$ of the p-Sylow subgroup $G(p)$ of G such that $\dim V(p)^H = N(H, G(p))$ for all $H \leq G(p)$. Let $\mathcal{V} = \bigotimes_{p \mid |G|} V(p)$. Then $\mathcal{V}$ is a representation of G and we will prove by induction on $|G|$ that $\mathcal{V}$ contains a subrepresentation R of G of dimension $N = N(e,G)$ such that $\dim R^H = N(H,G)$ for all p-subgroups H of G, all $p \in P$. So if $|K| < |G|$, $N(\ ,K)$ is a nonnegative integer valued function on the prime power subgroups of K and $W(p)$ is a representation of $K(p)$ realizing $N(\ ,K(p))$, we can assume $\mathcal{W} = \bigotimes_{p \mid |K|} W(p)$ contains a subrepresentation realizing $N(\ ,K)$.

Suppose that $N(\ ,G)$ is a nonnegative integer valued function defined on the prime power order subgroups of an Abelian group G satisfying conditions 1–5 and suppose $N(e,G) = N(H,G)$ for some $H \leq G(p)$, $|H| = p$. Then for any prime power order subgroup K of G, define $N(K/K \cap H, G/H) = N(K,G)$. It is clear that $N(\ ,G/H)$ satisfies conditions 1–5. Moreover, for any $K \leq G(p)$,

$$N(K,G) \overset{\text{def}}{=} N(K,G(p)) = N(KH,G(p)) \overset{\text{def}}{=} N(KH,G) \overset{\text{def}}{=} N(KH/H, G/H).$$

For by induction we can assume $N(K',G(p)) = N(K'H,G(p))$ for any $K' \nleq K$ and clearly we can assume $H \nleq K$. Select $K' < K$ such that $|K/K'| = p$ and use

condition 1 (Borel Formula) on $K' \leq KH$ to obtain $N(K, G(p)) = N(KH, G(p))$. It follows that in this case, a representation of G/H realizing $N(\ ,G/H)$ can be regarded a s an unfaithful representation of G (with kernel at least H) realizing $N(\ ,G)$.

Now for each $p \in \mathcal{P}$ and each $\mathbf{Z}_p \leq G$, we must have $N - N(\mathbf{Z}_p, G) > 0$, otherwise by the observation above we could assume we are given a dimension function on $G/\mathbf{Z}_p$. Of all the differences $N - N(\mathbf{Z}_p, G)$, $p \in \mathcal{P}$, let p_0 and $H_0 = \mathbf{Z}_{p_0}$ be such that $N - N(H_0, G)$ is a minimum. Then the representation $V(p_0)$ of $G(p_0)$ (the p_0-Sylow subgroup of G) contains an irreducible subrepresentation $W(p_0)$ of $G(p_0)$ on which H_0 acts without (nonzero) fixed points. For $q \neq p_0$ select $H = \mathbf{Z}_q \leq G(q)$ such that $N - N(H, G)$ is least for q (in general $N - N(H_0, G) \leq N - N(H, G)$) and let $W(q)$ be an irreducible subrepresentation of $V(q)$ on which H acts without fixed points. Then $\mathcal{W} = \bigotimes_{q \mid |G|} W(q)$ is a G-subrepresentation of $\mathcal{V} = \bigotimes_{q \mid |G|} V(q)$. Let R_1 be an irreducible G-subrepresentation of $\mathcal{W}$. If $|G|$ is larger than 2, R_1 has dimension 2, since R_1 induces a free, irreducible, real representation of a cyclic group of order larger than 2 (the cyclic group is $G/$kernel of $R_1 =$ kernel of $\mathcal{W}$).

Now R_1, is being a representation of G, has associated to it a dimension function $N_1(\ ,G)$ defined on *all* subgroups H of G by $N_1(H, G) = \dim R_1^H$. Set $\overline{N}_1(\ ,G) = N(\ ,G) - N_1(\ ,G)$. It is easy to verify that $\overline{N}_1(\ ,G)$ is a dimension function defined on the prime power subgroups of G satisfying conditions 1–5.

Since $\overline{N}_1(e, G) < N(e, G)$ and $\overline{N}_1(H_0, G) = N(H_0, G)$ we are presented with two situations: (a) $\overline{N}_1(e, G) = \overline{N}_1(H_0, G)$ or (b) $\overline{N}_1(e, G) > \overline{N}_1(H_0, G)$.

In case (a) the function $\overline{N}_1(\ ,G)$ may be replaced (as noted above) by a dimension function defined on the prime power subgroups of G/H_0. Since for any q-subgroup K of G, $\dim R_1^K = \dim W(q)^K$, the subrepresentation $W(q)^\perp$ of $V(q)$ realizes the dimension function $\overline{N}_1(\ ,G(q))$. Since $|G/H_0| < |G|$ by induction the tensor product of all the $W(q)^\perp$ contains a subrepresentation R^* of G/H_0 (which may be thought of as an unfaithful representation of G). R^* is a G-subrepresentation of $\mathcal{V}$, the tensor product of all the $V(q)$. $R^* \oplus R_1$ is the required representation in this case.

In (b), where we have $\overline{N}_1(e, G) > \overline{N}_1(H_0, G)$, note that $\overline{N}_1(e, G) - \overline{N}_1(H_0, G)$ is still a minimum of all differences $\overline{N}_1(e, G) - \overline{N}_1(H, G)$. Since the function $\overline{N}_1(\ ,G(Q))$ is realized by the subrepresentation $W(q)^\perp$ of $V(q)$, we can repeat the procedure again obtaining another irreducible subrepresentation R_2 or G with an associated dimension function $N_2(\ ,G)$ defined on the prime power subgroups of G (it is the restriction of a dimension function defined on all subgroups of G). Letting $\overline{N}_2(\ ,G) = \overline{N}_1(\ ,G) = N_2(\ ,G)$ we again have a dimension function satisfying conditions 1–5 and we proceed as above. Eventually we obtain a dimension function $\overline{N}_k(\ ,G)$ such that $\overline{N}_k(e, G) = \overline{N}_k(H_0, G)$ ($k = N - N(H_0, G)$). By case (a) and induction there is a G-subrepresentation of $\mathcal{V}$, R^* realizing $\overline{N}_k(\ ,G)$. The representation $R = R^* \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_k$ is the required G-subrepresentation of $\mathcal{V}$.

Now suppose $\overline{R}$ is another G-subrepresentation such that for any prime power order subgroup H of G, $\dim \overline{R}^H = N(H, G)$. Let H be an arbitrary subgroup of G and by induction assume $\dim \overline{R}^K - \dim R^k$ is even for all subgroups K of G with $|K| < |H|$. Select $K \leq H$ so that $|H/K|$ is an odd prime p (if this is not possible

then H is a 2-group and $\dim \overline{R}^H - \dim R^H$ is zero). The group $H/K = \mathbf{Z}_p$ acts on both $\overline{R}^K$ and R^K. It follows that both $\dim \overline{R}^K - \dim \overline{R}^H$ and $\dim R^K - \dim R^H$ are even and therefore $\dim \overline{R}^H - \dim R^H$ is even. This completes the proof of the theorem. □

2. Proof of the corollary. Suppose G is a finite Abelian group with cyclic 2-Sylow subgroup, $G(2)$, and suppose $N(\ ,G)$ is a nonnegative integer valued function defined on the p-subgroups of G, $p \mid |G|$, satisfying conditions 1–4. By condition 4, for any proper subgroup H of $G(2)$, $N - N(H,G)$ is even. Suppose that $N - N(G(2),G)$ is odd. For each $p \mid |G|$, let $\overline{V}(p) = V(p) \oplus 1$, where 1 denotes the trivial one-dimensional representation of $G(p)$. It is easy to see that $\overline{V}(p)$ realizes $N(\ ,G(p)) + 1$. The function $N^*(\ ,G) = N(\ ,G) + 1$ corresponds to the G-action on the unreduced suspension on X.

Now since $N - N(G(2),G)$ is odd, $\overline{V}(2)$ has an irreducible summand of dimension 1 on which H, the maximal proper subgroup of $G(2)$, acts trivially and on which $G(2)$ acts nontrivially. Denote this summand by $W(2)$ for any $p \neq 2$ let $W(p)$ be a one-dimensinal trivial subrepresentation of $\overline{V}(p)$. Then $R_1 \bigotimes_{p \mid |G|} W(p)$ is a 1-dimensional G-representation with a very large kernel and is a subrepresentation of $\overline{V} = \bigotimes_{p \mid |G|} \overline{V}(p)$. Let $N_1(\ ,G)$ be the dimension function associated with R_1 ($N_1(\ ,G)$ is actually defined on all subgroups of G). Setting $\overline{N}(\ ,G) = N^*(\ ,G) - N_1(\ ,G)$ we see that $N - \overline{N}(H,G)$ is now even for *all* prime power subgroups of G so $\overline{N}(\ ,G)$ satisfies conditions 1–5. By the argument §1, $\overline{N}(\ ,G)$ is realized by a subrepresentation R of $\bigotimes_{p \mid |G|} W(P)^\perp$, since $\overline{N}(\ ,G(p))$ is realized by $W(p)^\perp$ for all $p \mid |G|$. Then $R \oplus R_1$ is a subrepresentation of the G-representation $\overline{V}$ which realizes $N^*(\ ,G)$. This establishes the corollary.

EXAMPLES. Let $G = \mathbf{Z}_6$, $N = 2$, $N(\mathbf{Z}_2,G) = 1$, $N(\mathbf{Z}_3,G) = 0$. If we "suspend" $N(\ ,G)$ we have $N^* = 3$, $N^*(\mathbf{Z}_2,G) = 2$, $N^*(\mathbf{Z}_3,G) = 1$. Then the construction of §§2 and 1 yeilds the 3-dimensional representation of G, given on a generator by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & & \\ 0 & & R(2\pi/3) \end{pmatrix}$$

where $R(2\pi/3)$ is a 2×2 rotation matrix.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF MISSOURI–ST. LOUIS, ST. LOUIS, MISSOURI 63121