ORIENTATION PRESERVING ACTIONS
OF FINITE ABELIAN GROUPS ON SPHERES
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ABSTRACT. If $G$ is a finite Abelian group acting as a $\mathbb{Z}(p)$-homology $n$-sphere $X$ (where $\mathcal{P}$ is the set of primes dividing $|G|$), then there is an integer valued function $n(\ ,G)$ defined on the prime power subgroups $H$ of $G$ such that $X^H$ has the $\mathbb{Z}(p)$-homology of a sphere $S^{n(H,G)}$. We prove here that there exists a real representation $R$ of $G$ such that for any prime power subgroup $H$ of $G$, $\dim(S(R^H)) = n(H,G)$ where $S(R^H)$ is the unit sphere of $R^H$, provided that $n - n(H,G)$ is even whenever $H$ is a 2-subgroup of $G$.

0. Introduction. Suppose that $G$ is a finite Abelian group and let $\mathcal{P}$ be the set of primes dividing $|G|$. If $G$ acts on a finite CW-complex $X$ which has the $\mathbb{Z}(p)$-homology of $S^n$, then for any $p \in \mathcal{P}$ and $p$-subgroup $H$ of $G$ the fixed point set of $H$ on $X$, $X^H$, has the $\mathbb{Z}(p)$-homology of $S^{n(H,G)}$ for some integer $n(H,G) \geq -1$ ($-1$ signifies empty). This is a well-known consequence of Smith theory ([2, III or 1, IV], e.g.). Thus we obtain in this way an integer valued function, $n(\ ,G)$, defined on the set of $p$-subgroups of $G$ by $H \mapsto n(H,G)$ (note that $n(e,G) = n$). This function is called the “dimension function” and has a considerable literature (see [5, 3, 6, 7, 9]; [10] gives a related extensive bibliography).

The function $n(\ ,G)$ satisfies the following well-known conditions (see [1, XIII, 2.3; IV, 4.4, 4.7]):

1. (Borel Formula) If $H \leq K$ are both $p$-subgroups of $G$ and $K/H = \mathbb{Z}_p + \mathbb{Z}_p$, then $n(H,G) - n(K,G) = \sum(n(K'/G) - n(K,G))$ with the sum over all $H \leq K' \leq K$ such that $K'/H = \mathbb{Z}_p$.

2. If $H \leq K$ are $p$-subgroups of $G$, then $n(K,G) \leq n(H,G)$.

3. If $H \leq K$ are $p$-subgroups of $G$ with $K/H = \mathbb{Z}_p$ and $p$ odd, then $n(H,G) - n(K,G)$ is even.

4. If $H \leq K' \leq K$ are 2-subgroups of $G$ such that $K/H = \mathbb{Z}_4$, $K'/H = \mathbb{Z}_2$, then $n(H,G) - n(K',G)$ is even.

For each $p \in \mathcal{P}$, let $G(p)$ denote the $p$-Sylow subgroup of $G$ and set $N(\ ,G) = n(\ ,G) + 1$. The function $N(\ ,G)$ restricted to the subgroups of $G(p)$ will naturally be denoted by $N(\ ,G(p))$. In [8] it was shown that $N(\ ,G(p))$ is realized by a real representation $V(p)$ of $G(p)$ which means that for each $H \leq G(p)$, $\dim V(p)^H = N(H,G(p)) = N(H,G)$. If $S(V(p))$ denotes the unit sphere of $V(p)$, then $\dim S(V(p)^H) = n(H,G)$.

Here we are interested in the existence of a real representation $R$ of the Abelian group $G$ such that for any $p$-subgroup $H$ of $G$ (for any $p \in \mathcal{P}$), $\dim R^H = N(H,G)$. Thus $R$ would be a simultaneous realization of the functions $N(\ ,G(p))$, $p \in \mathcal{P}$.

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It should be noted that in the special case where $X^H$ is a homology sphere for all $H \leq G$, then $N(\ ,G)$ is defined on all subgroups of $G$ and tom Dieck has shown in [3] that $N(\ ,G)$ is realized by a difference of representations. In general, this is best possible.

Clearly $N(\ ,G)$ satisfies conditions 1–4 if $n(\ ,G)$ does. We will denote $N(e,G)$ by $N$. As an example to show that some condition beyond 1–4 is needed, suppose $G = \mathbb{Z}_6$, $N = 2$, $N(\mathbb{Z}_3,G) = 0$, and $N(\mathbb{Z}_2,G) = 1$. Then $N(\ ,G)$ satisfies conditions 1–4 but there is no real representation of $G$ which realizes these numbers simultaneously as dimensions. From now on we will assume the following “orientation preserving” condition holds, in addition to conditions 1–4:

5. If $H$ is any 2-subgroup of $G$, then $N - N(H,G)$ is even.

We obtain the following theorem and corollary:

**Theorem.** Let $G$ be a finite Abelian group and suppose $N(\ ,G)$ is a nonnegative integer valued function defined on the $p$-subgroups of $G$ for all $p \mid |G|$, satisfying conditions 1–5. Then there exists a real representation $R$ of $G$ such that for any $p$-subgroup $H$ of $G$, $p \mid |G|$, $\dim RH = N(H,G)$. Furthermore, if $\bar{R}$ is another such representation of $G$ then for all subgroups $H$ of $G$,

$$\dim RH \equiv \dim \bar{R}^H \pmod{2}.$$ 

**Corollary.** Let $G$ be a finite Abelian group and suppose the 2-Sylow subgroup of $G$ is cyclic. If $N(\ ,G)$ is a nonnegative integer valued function defined on the $p$-subgroups of $G$, $p \mid |G|$ satisfying only conditions 1–4, then there exists a real representation $R$ such that for any $p$-subgroup $H$ of $G$, $p \mid |G|$, $\dim RH = N(H,G) + 1$.

In §§1 and 2 we prove the theorem and corollary respectively. We thank the referee for several suggestions leading to an improved exposition.

**1. Proof of the theorem.** Let $G$ be a finite Abelian group and suppose $N(\ ,G)$ is a nonnegative integer valued function on the $p$-subgroups of $G$, for all $p \in \mathbb{P}$, satisfying conditions 1–5. By [8], for each $p \in \mathbb{P}$ there is a representation $V(p)$ of the $p$-Sylow subgroup $G(p)$ of $G$ such that $\dim V(p)^H = N(H,G(p))$ for all $H \leq G(p)$. Let $\mathcal{V} = \bigotimes_{p \mid |G|} V(p)$. Then $\mathcal{V}$ is a representation of $G$ and we will prove by induction on $|G|$ that $\mathcal{V}$ contains a subrepresentation $R$ of $G$ of dimension $N = N(e,G)$ such that $\dim RH = N(H,G)$ for all $p$-subgroups $H$ of $G$, all $p \in \mathbb{P}$. So if $|K| < |G|$, $N(\ ,K)$ is a nonnegative integer valued function on the prime power subgroups of $K$ and $W(p)$ is a representation of $K(p)$ realizing $N(\ ,K(p))$, we can assume $W = \bigotimes_{p \mid |K|} W(p)$ contains a subrepresentation realizing $N(\ ,K)$.

Suppose that $N(\ ,G)$ is a nonnegative integer valued function defined on the prime power order subgroups of an Abelian group $G$ satisfying conditions 1–5 and suppose $N(e,G) = N(H,G)$ for some $H \leq G(p)$, $|H| = p$. Then for any prime power order subgroup $K$ of $G$, define $N(K/K \cap H, G/H) = N(K,G)$. It is clear that $N(\ ,G/H)$ satisfies conditions 1–5. Moreover, for any $K \leq G(p)$,

$$N(K,G) \overset{\text{def}}{=} N(K,G(p)) = N(KH,G(p)) \overset{\text{def}}{=} N(KH,G) \overset{\text{def}}{=} N(KH/H,G/H).$$

For by induction we can assume $N(K',G(p)) = N(K'H,G(p))$ for any $K' \nleq K$ and clearly we can assume $H \nleq K$. Select $K' < K$ such that $|K/K'| = p$ and use
condition 1 (Borel Formula) on $K' \leq KH$ to obtain $N(K, G(p)) = N(KH, G(p))$. It follows that in this case, a representation of $G/H$ realizing $N(\cdot, G/H)$ can be regarded as an unfaithful representation of $G$ (with kernel at least $H$) realizing $N(\cdot, G)$.

Now for each $p \in P$ and each $Z_p \leq G$, we must have $N - N(Z_p, G) > 0$, otherwise by the observation above we could assume we are given a dimension function on $G/Z_p$. Of all the differences $N - N(Z_p, G)$, $p \in P$, let $p_0$ and $H_0 = Z_{p_0}$ be such that $N - N(H_0, G)$ is a minimum. Then the representation $V(p_0)$ of $G(p_0)$ (the $p_0$-Sylow subgroup of $G$) contains an irreducible subrepresentation $W(p_0)$ of $G(p_0)$ on which $H_0$ acts without (nonzero) fixed points. For $q \neq p_0$ select $H = Z_q \leq G(q)$ such that $N - N(H, G)$ is least for $q$ (in general $N - N(H_0, G) \leq N - N(H, G)$) and let $W(q)$ be an irreducible subrepresentation of $V(q)$ on which $H$ acts without fixed points. Then $W = \bigotimes_q |G| W(q)$ is a $G$-subrepresentation of $V = \bigotimes_q |G| V(q)$.

Let $R_1$ be an irreducible $G$-subrepresentation of $W$. If $|G|$ is larger than 2, $R_1$ has dimension 2, since $R_1$ induces a free, irreducible, real representation of a cyclic group of order larger than 2 (the cyclic group is $G$/kernel of $R_1$ = kernel of $W$).

Now $R_1$ is being a representation of $G$, has associated to it a dimension function $N_1(\cdot, G)$ defined on all subgroups $H$ of $G$ by $N_1(H, G) = \dim R_1^H$. Set $\overline{N}_1(G) = N(\cdot, G) - N_1(\cdot, G)$. It is easy to verify that $\overline{N}_1(\cdot, G)$ is a dimension function defined on the prime power subgroups of $G$ satisfying conditions 1–5.

Since $\overline{N}_1(e, G) < N(e, G)$ and $\overline{N}_1(H_0, G) = N(H_0, G)$ we are presented with two situations: (a) $\overline{N}_1(e, G) = \overline{N}_1(H_0, G)$ or (b) $\overline{N}_1(e, G) > \overline{N}_1(H_0, G)$.

In case (a) the function $\overline{N}_1(\cdot, G)$ may be replaced (as noted above) by a dimension function defined on the prime power subgroups of $G/H_0$. Since for any $q$-subgroup $K$ of $G$, $\dim R_1^K = \dim W(q)^K$, the subrepresentation $W(q)^\perp$ of $V(q)$ realizes the dimension function $\overline{N}_1(\cdot, G(q))$. Since $|G/H_0| < |G|$ by induction the tensor product of all the $W(q)^\perp$ contains a subrepresentation $R^*$ of $G/H_0$ (which may be thought of as an unfaithful representation of $G$). $R^*$ is a $G$-subrepresentation of $V$, the tensor product of all the $V(q)$. $R^* \oplus R_1$ is the required representation in this case.

In (b), where we have $\overline{N}_1(e, G) > \overline{N}_1(H_0, G)$, note that $\overline{N}_1(e, G) - \overline{N}_1(H_0, G)$ is still a minimum of all differences $\overline{N}_1(e, G) - \overline{N}_1(H, G)$. Since the function $\overline{N}_1(\cdot, G(Q))$ is realized by the subrepresentation $W(q)^\perp$ of $V(q)$, we can repeat the procedure again obtaining another irreducible subrepresentation $R_2$ or $G$ with an associated dimension function $N_2(\cdot, G)$ defined on the prime power subgroups of $G$ (it is the restriction of a dimension function defined on all subgroups of $G$). Letting $\overline{N}_2(\cdot, G) = \overline{N}_1(\cdot, G) = N_2(\cdot, G)$ we again have a dimension function satisfying conditions 1–5 and we proceed as above. Eventually we obtain a dimension function $\overline{N}_k(\cdot, G)$ such that $\overline{N}_k(e, G) = \overline{N}_k(H_0, G)$ ($k = N - N(H_0, G)$). By case (a) and induction there is a $G$-subrepresentation of $\overline{V}$, $R^*$ realizing $\overline{N}_k(\cdot, G)$. The representation $R = R^* \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_k$ is the required $G$-subrepresentation of $\overline{V}$.

Now suppose $R$ is another $G$-subrepresentation such that for any prime power order subgroup $H$ of $G$, $\dim R^H = N(H, G)$. Let $H$ be an arbitrary subgroup of $G$ and by induction assume $\dim \overline{R}_K^H - \dim R^k$ is even for all subgroups $K$ of $G$ with $|K| < |H|$. Select $K \leq H$ so that $|H/K|$ is an odd prime $p$ (if this is not possible
then \( H \) is a 2-group and \( \dim \overline{R}^H - \dim R^H \) is zero). The group \( H/K = \mathbb{Z}_p \) acts on both \( \overline{R}^K \) and \( R^K \). It follows that both \( \dim \overline{R}^K - \dim \overline{R}^H \) and \( \dim R^K - \dim R^H \) are even and therefore \( \dim \overline{R}^H - \dim R^H \) is even. This completes the proof of the theorem. □

2. Proof of the corollary. Suppose \( G \) is a finite Abelian group with cyclic 2-Sylow subgroup, \( G(2) \), and suppose \( N(\ ,G) \) is a nonnegative integer valued function defined on the \( p \)-subgroups of \( G \), \( p \mid |G| \), satisfying conditions 1–4. By condition 4, for any proper subgroup \( H \) of \( G(2) \), \( N - N(H,G) \) is even. Suppose that \( N - N(G(2),G) \) is odd. Let \( V(2) \) realize \( N(\ ,G(2)) + 1 \). The function \( N^*(\ ,G) = N(\ ,G) + 1 \) corresponds to the \( G \)-action on the unreduced suspension on \( X \).

Now since \( N - N(G(2),G) \) is odd, \( V(2) \) has an irreducible summand of dimension 1 on which \( H \), the maximal proper subgroup of \( G(2) \), acts trivially and on which \( G(2) \) acts nontrivially. Denote this summand by \( W(2) \) for any \( p \neq 2 \) let \( W(p) \) be a one-dimensional trivial subrepresentation of \( V(p) \). Then \( R_1 \otimes_{p\mid|G|} W(p) \) is a 1-dimensional \( G \)-representation with a very large kernel and is a subrepresentation of \( \overline{V} = \bigotimes_{p\mid|G|} V(p) \). Let \( N_1(\ ,G) \) be the dimension function associated with \( R_1 \) \((N_1(\ ,G) \) is actually defined on all subgroups of \( G \)). Setting \( \overline{N}(\ ,G) = N^*(\ ,G) - N_1(\ ,G) \) we see that \( N - \overline{N}(H,G) \) is now even for all prime power subgroups of \( G \) so \( \overline{N}(\ ,G) \) satisfies conditions 1–5. By the argument §1, \( \overline{N}(\ ,G) \) is realized by a subrepresentation \( R \) of \( \bigotimes_{p\mid|G|} W(P)^\perp \), since \( \overline{N}(\ ,G(p)) \) is realized by \( W(p)^\perp \) for all \( p \mid |G| \). Then \( R \oplus R_1 \) is a subrepresentation of the \( G \)-representation \( \overline{V} \) which realizes \( N^*(\ ,G) \). This establishes the corollary.

EXAMPLES. Let \( G = \mathbb{Z}_6 \), \( N = 2 \), \( N(\mathbb{Z}_2, G) = 1 \), \( N(\mathbb{Z}_3, G) = 0 \). If we “suspend” \( N(\ ,G) \) we have \( N^* = 3 \), \( N^*(\mathbb{Z}_2, G) = 2 \), \( N^*(\mathbb{Z}_3, G) = 1 \). Then the construction of §§2 and 1 yeilds the 3-dimensional representation of \( G \), given on a generator by

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & R(2\pi/3)
\end{pmatrix}
\]

where \( R(2\pi/3) \) is a 2 \times 2 rotation matrix.

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