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SOME HOMOTOPY PROPERTIES OF THE HOMEOMORPHISM GROUPS OF $R^\infty(Q^\infty)$-MANIFOLDS

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ABSTRACT. In this note we will prove that, given an $R^\infty(Q^\infty)$-manifold M, there is a deformation of Homo(M) into Homeo(M) whose final stage is a weak homotopy equivalence, and that if M has the homotopy type of a finite simplicial complex, then Homeo(M) is an ANE($\mathcal{CW}(\mathcal{M})$) and an ANE($\mathcal{CW}(\mathcal{C}), G_\delta$).

0. Introduction. All spaces are Hausdorff and maps are continuous. The composition of two maps $f\colon A \to B$ and $g\colon B \to C$ will be denoted by gf. Let $\mathcal{CG}$ denote the category whose objects are compactly generated spaces (k-spaces) and whose morphisms are continuous maps [**Gr**, 8.1]. Given two spaces X and Y, following [**Gr**], let $X \times Y$ denote the product of X and Y in $\mathcal{CG}$, let $X \times_c Y$ be the Cartesian product, let $C(X,Y)$ be the space of maps from X to Y with the compact-open topology, and let $Y^X = kC(X,Y)$ [**Gr**, 8.14]. Similarly, let Homeo(X) (Homo(X), resp.) denote the space of homeomorphisms (homotopy equivalences, resp.) of X with compact-open topology as a subspace of $C(X,X)$. Recall that kHomeo(X) and kHomo(X) are subspaces of X^X in the sense of [**Hy**, p. 201]. Following [$\mathbf{L_3}$], let $\mathcal{CW}(\mathcal{C})$ and $\mathcal{CW}(\mathcal{M})$ denote the classes of pseudo CW-complexes generated by the class $\mathcal{C}$ of Hausdorff compact spaces and by the class $\mathcal{M}$ of metric spaces, respectively. Let $E^\infty = \operatorname{dirlim} E^n$, where E^n is either the n-Euclidean space or the n-fold product of the Hilbert cube Q. By an *E^∞-manifold*, we mean a separable paracompact space which is locally homeomorphic to E^∞.

In this note we will prove that given an E^∞-manifold M, there is a deformation of Homo(M) in itself into Homeo(M) whose final stage is a weak homotopy equivalence (this partially answers a question raised by P. J. Kahn some time ago [$\mathbf{L_3}$, §0]), and that if M has the homotopy type of a finite complex, then Homeo(M) is an absolute neighborhood extensor [**Hu**] (abbreviated ANE) for the class $\mathcal{CW}(\mathcal{M})$ and is an ANE($\mathcal{CW}(\mathcal{C}), G_\delta$). By a space $S \in$ ANE($\mathcal{X}, G_\delta$), we mean that if $X \in \mathcal{X}$ and A is a closed G_δ-subset of X, then every map $f\colon A \to S$ has a continuous extension over a neighborhood of A in X.

Throughout this note $I = [0,1]$. By an *E^∞-manifold fiber bundle*, we mean a locally trivial bundle in $\mathcal{CG}$, $p\colon X \to B$, whose fiber is an E^∞-manifold. If (X,p,B) is a bundle, W a subset of X, and A a subset of B, we write $W_A = W \cap p^{-1}(A)$; specially, $X_A = p^{-1}(A)$. Let (X,p,B) and (Y,p',B) be bundles and W a subset of X. A map $f\colon W \to Y$ is said to be *fiber preserving* (f.p.) if $p'f = p|W$. If $f\colon W \to B \times Z$ is an f.p. map, for each $b \in B$ let $f_b\colon W_{\{b\}} \to Z$ denote the map $p_Z f|W_{\{b\}}$ where $p_Z\colon B \times Z \to Z$ is the projection. Given an $R^\infty(Q^\infty)$-manifold M,

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following [$\mathbf{L_1}$, $\mathbf{L_2}$], we write $M = \operatorname{dirlim} M_n$, where M_n is a compact submanifold (compact Q-submanifold) of M_{n+1} for each $n = 1, 2, \ldots$.

Given a map $f\colon B \to Y^X$ or $f\colon B \to C(X,Y)$, by the *associate* of f we mean the f.p. function $\hat{f}\colon B \times Y \to B \times Y$ defined by $\hat{f}(b,x) = (b, f(b)(x))$, and vice versa; we also call f the associate of $\hat{f}$ [**D**, p. 261]. By [**Gr**, 8.17], the continuity of f and that of its associate are equivalent if $B, X, Y \in \mathcal{CG}$. Given a subset A of X, let i_A denote the inclusion $A \to X$ and id_A the identity map on A.

1. A canonical deformation. The main result of this section is Theorem 1.3 which proves the existence of a deformation of the space of homotopy equivalences into the space of homeomorphisms. The proof of the following lemma is straightforward.

LEMMA 1.1. *Let X, Y, and Z be topological spaces. Let*

$$F\colon C(X,Y) \to C(X \times_c Z, Y \times_c Z)$$

be the map defined by $F(g) = g \times \mathrm{id}_Z$. Then F is an embedding; i.e., $C(X,Y)$ is homeomorphic to the subspace $F(C(X,Y))$ of $C(X \times_c Z, Y \times_c Z)$. □

We need another lemma for the proof of Theorem 1.3. Recall that $X \times_c I \cong X \times I$ for $X \in \mathcal{CG}$ [**Gr**, 8.11].

LEMMA 1.2. *Let M, N be topological spaces. If $g\colon M \times_c I \to M$ and $h\colon N \times_c I \to N$ are given homotopies, then g and h induce the homotopies G and H on $C(M,N)$ defined by $G(f,t) = fg_t$ and $H(f,t) = h_t f$ for all $f \in C(M,N)$ and $t \in I$.*

PROOF. For the continuity of G, consider the following composition:

$$\overline{g}\colon C(M,N) \xrightarrow{\tilde{p}_M} C(M \times_c I, N) \xrightarrow{\tilde{g}'} C(M \times_c I, N) \xrightarrow{\sigma} C(I, C(M,N)),$$

where $\tilde{p}_M, \tilde{g}'$ are maps naturally induced [**D**, p. 259] from the projections $p_M\colon M \times_c I \to M$ and $g'\colon M \times_c I \to M \times_c I$ defined by $g'(x,t) = (g(x,t), t)$, and where σ is the embedding defined by $\sigma(f)(t)(x) = f(x,t)$ for all $t \in I$ and $x \in M$ [**J**, Theorem 1.1]. Define

$$G\colon C(M,N) \times_c I \xrightarrow{\overline{g} \times \mathrm{id}_I} C(I, C(M,N)) \times_c I \xrightarrow{e} C(M,N),$$

where e is the evaluation map. Then G is a homotopy. Let $f \in C(M,N)$ and $t \in I$. We have

$$\begin{aligned} G(f,t) &= e(\overline{g} \times \mathrm{id}_I)(f,t) = e((\sigma \tilde{g}' \tilde{p}_M) \times \mathrm{id}_I)(f,t) \\ &= e(\sigma \tilde{g}' \tilde{p}_M(f), t) = e(\sigma(f p_M g'), t) = \sigma(f p_M g')(t) \\ &= f p_M g'(\cdot, t) = f g(\cdot, t) = f g_t. \end{aligned}$$

For the continuity of H, consider the composition

$$\overline{h}\colon C(M,N) \xrightarrow{F} C(M \times_c I, N \times_c I) \xrightarrow{\tilde{h}} C(M \times_c I, N) \xrightarrow{\sigma} C(I, C(M,N)),$$

where the embedding F is given by Lemma 1.1, $\tilde{h}$ is induced from h and σ is from [**J**, Theorem 1.1] as above. Define

$$H\colon C(M,N) \times_c I \xrightarrow{\overline{h} \times \mathrm{id}_I} C(I, C(M,N)) \times_c I \xrightarrow{e} C(M,N).$$

Then H is continuous. Finally, let $f \in C(M,N)$ and $t \in I$. We have

$$\begin{aligned} H(f,t) &= e(\overline{h} \times \mathrm{id}_I)(f,t) = e(\overline{h}(f),t) = \overline{h}(f)(t) \\ &= (\sigma \tilde{h} F(f))(t) = \tilde{h}(F(f))(\cdot,t) \\ &= (\tilde{h}(f \times \mathrm{id}_I))(\cdot,t) = h((f \times \mathrm{id}_I)(\cdot,t)) \\ &= h(f(\cdot),t) = h_t f. \quad \square \end{aligned}$$

REMARK 1. In Lemma 1.2, if M and N are k-spaces, then G and H are also homotopies on $N^M = kC(M,N)$ (by use of [**Gr**, 8.8(vi)]).

THEOREM 1.3. *Let M be an E^∞-manifold. Then, there is a deformation of* $\mathrm{Homo}(M)$ *in itself into* $\mathrm{Homeo}(M)$ *whose final map* $r\colon \mathrm{Homo}(M) \to \mathrm{Homeo}(M)$ *is a weak homotopy equivalence.*

PROOF. To avoid confusion, we let N be another copy of M and consider $\mathrm{Homo}(M,N)$ and $\mathrm{Homeo}(M,N)$ instead of $\mathrm{Homo}(M)$ and $\mathrm{Homeo}(M)$, respectively. By [**He**, Theorem C], we can identify M and N with $K \times E^\infty$ and $L \times E^\infty$, respectively, where K and L are locally finite simplicial complexes. We also identify K and L with $K \times 0$ and $L \times 0$, respectively. Let $g\colon M \times I \to M$ and $h\colon N \times I \to N$ be natural strong deformation retractions of M and N onto K and L, respectively, with $g_1 = p_K$ and $h_1 = p_L$ (the projections). Let G and H be the homotopies on $C(M,N)$ induced by g and h given by Lemma 1.2. Define $D_t\colon C(M,N) \to C(M,N)$ by

$$D_t = \begin{cases} H_{4t} & \text{if } t \in [0,1/4], \\ G_{4t-1} & \text{if } t \in [1/4,1/2], \\ G^*_{4t-2} & \text{if } t \in [1/2,3/4], \\ H^*_{4t-3} & \text{if } t \in [3/4,1], \end{cases}$$

where $G^*_t = G_{1-t}$ and $H^*_t = H_{1-t}$. Then

$$D\colon f \simeq p_L f \simeq p_L f p_K \simeq p_L((p_L f|K) \times \mathrm{id}_{E^\infty}) \simeq (p_L f|K) \times \mathrm{id}_{E^\infty}, w3$$

and

$$D_1(C(M,N)) = \{q \times \mathrm{id}_{E^\infty} \,|q \in C(K,L)\} \cong C(K,L)$$

(by Lemma 1.1) which is a metrizable space [**Ku**, Theorem 1, p. 93].

Observe that $D(\mathrm{Homo}(M,N) \times_c I) \subset \mathrm{Homo}(M,N)$. So, D deforms $\mathrm{Homo}(M,N)$ in itself into the metrizable subspace $B \equiv \mathrm{Homo}(M,N) \cap \{q \times \mathrm{id}_{E^\infty} \,|q \in C(K,L)\}$. Recall that since B is metrizable, it also is a subspace of $k\,\mathrm{Homo}(M,N)$. Now, we would like to deform B into $\mathrm{Homeo}(M,N)$. Consider the associate g of the inclusion $i_B\colon B \to \mathrm{Homo}(M,N)$, i.e.,

$$g\colon B \times K \times E^\infty \to B \times L \times E^\infty$$

defined by $g(b,x,y) = (b, q(x), y)$, where $b = q \times \mathrm{id}_{E^\infty}$. Then, g is continuous [**Gr**, 8.17], and g is an f.p. homotopy equivalence [**Ja**, Proposition 7.58]. Therefore, by [$\mathbf{L_3}$, Lemma 3.3], g is f.p. homotopic to an f.p. homeomorphism, say $\overline{g}$. This homotopy induces a homotopy $R\colon B \times I \to \mathrm{Homo}(M,N)$ with $R_0 = i_B$ and $R_1(B) \subset \mathrm{Homeo}(M,N)$. Combining D and R, we will obtain a desired deformation with a final map $r = R_1 D_1$.

Finally, let $i\colon \mathrm{Homeo}(M) \to \mathrm{Homo}(M)$ be the natural inclusion. Then we have $ir = r \simeq \mathrm{id}_{\mathrm{Homo}(M)}$. By [$\mathbf{L_3}$, Lemma 3.3], i is a weak homotopy equivalence (i.e., i

induces isomorphisms on homotopy groups). It follows easily that r is also a weak homotopy equivalence. $\square$

To conclude this section, we prove the following proposition which will be used in the next section. In fact, the proposition indicates that $\mathrm{Homeo}(M)$ is not a closed subset of $\mathrm{Homo}(M)$. Therefore, it is interesting to know whether the deformation in the above theorem can be chosen such that $\mathrm{Homeo}(M)$ is deformed in itself.

PROPOSITION 1.4. *Let A be a closed subset (G_δ-subset, resp.) of $B \in \mathcal{CW}(\mathcal{M})$ ($B \in \mathcal{CW}(\mathcal{C})$, resp.), and (X, p, B) an E^∞-manifold fiber bundle. If $f\colon X \to X$ is an f.p. homotopy equivalence with $f|X_A$ a continuous injection, then there is an f.p. map $h\colon X \to X$ such that $h|X_{B-A}\colon X_{B-A} \to X_{B-A}$ is an f.p. homeomorphism and that f is f.p. homotopic to h $(\mathrm{rel}\, X_A)$.*

PROOF. Similar to the proof of [$\mathbf{L_3}$, Theorem 3.4], we will assume that $X = B \times M$, where M is an E^∞-manifold. Since A is a closed G_δ-set in B, we can write $B - A = \bigcup\{C_n | n = 1, 2, \dots\}$, where each C_n is a closed G_δ-set in B and $C_n \subset$ (interior of C_{n+1}). Write $M = \mathrm{dirlim}\, M_n$ and assume $C_0 = \varnothing$ and $M_0 = \varnothing$. We now define by induction a sequence of f.p. homotopies $H^n\colon B \times M \times I \to B \times M$ and $h_n = H_1^n$ such that

(i)$_n$ $H^n\colon h_{n-1} \simeq h_n$ (f.p.) $\mathrm{rel}(A \times M) \cup (C_{n-1} \times M) \cup (B \times M_{n-1})$,

(ii)$_n$ $h_n|C_n \times M\colon C_n \times M \to C_n \times M$ is an f.p. homeomorphism, and

(iii)$_n$ $h_n|B \times M_n\colon B \times M_n \to B \times M$ is an f.p. embedding.

First, since $f|A \times M_1$ is an f.p. embedding, there is from [$\mathbf{L_3}$, Lemma 3.3] an f.p. homotopy $F\colon B \times M \times I \to B \times M$ $(\mathrm{rel}\, A \times M_1)$ such that $F_0 = f$ and F_1 is an f.p. homeomorphism. Let $s\colon B \to I$ be a map with $s^{-1}(0) = A$ and $s^{-1}(1) = C_1$. Define $H^1\colon B \times M \times I \to B \times M$ by $H^1(b, x, t) = F(b, x, s(b)t)$. Then we have

(i)$_1$ $H^1\colon f\ (\equiv h_0) \simeq h_1$ (f.p.) $\mathrm{rel}\, A \times M$,

(ii)$_1$ $h_1|C_1 \times M = F_1|C_1 \times M$ is an f.p. homeomorphism, and

(iii)$_1$ $h_1|B \times M_1\colon B \times M_1 \to B \times M$ is an f.p. embedding, by use of [$\mathbf{L_3}$, Lemma 1.2].

Second, assume that H^{n-1} has been defined such that (i)$_{n-1}$, (ii)$_{n-1}$, and (iii)$_{n-1}$ are satisfied. By [$\mathbf{L_3}$, Lemma 3.3], there is an f.p. homotopy $G\colon B \times M \times I \to B \times M$ $(\mathrm{rel}\, A \times M_n \cup C_{n-1} \times M \cup B \times M_{n-1})$ such that $G_0 = h_{n-1}$ and G_1 is an f.p. homeomorphism. Similar to above, by use of an Urysohn function and [$\mathbf{L_3}$, Lemma 1.2], we can obtain a homotopy H^n such that (i)$_n$, (ii)$_n$, and (iii)$_n$ are satisfied.

Finally, define $h = \lim h_n$; then, h is well defined and continuous by use of (iii)$_n$; and, $h|(B - A) \times M$ is an f.p. homeomorphism by use of (ii)$_n$ and the condition $C_n \subset$ (interior of C_{n+1}). Moreover, a desired homotopy $H\colon B \times M \times I \to B \times M$ from f to h can be defined by

$$H(b, x, t) = \begin{cases} H^n(b, x, 2^n(t-1) + 2) & \text{if } t \in J_n, \\ h(x) & \text{if } t = 1, \end{cases}$$

where $J_n = [1 - (1/2^{n-1}), 1 - (1/2^n)]$. Then, H is well defined and continuous by (i)$_n$. $\square$

2. Absolute neighborhood extensor properties of $\mathrm{Homeo}(M)$**.** We first prove some lemmas that will be used in the proof of the main result of this section, Theorem 2.3.

LEMMA 2.1. *Let K be a compact metric space and M an E^∞-manifold. Then the space $M^K \in \mathcal{CW}(\mathcal{M})$; moreover, it is an* ANE($\mathcal{M}$) *and an* ANE($\mathcal{C}$). *Consequently, M^K is an* ANE($\mathcal{CW}(\mathcal{M})$) *and an* ANE($\mathcal{CW}(\mathcal{C})$), *and so is $C(K, M)$.*

PROOF. The proof is implicit in that of [**Hy**, Theorem 8.2]. Because of its simplicity in this case, we will give its outline.

Write $M = \operatorname{dirlim} M_n$, where each M_n is a compact ANR metric space with $M_n \subset M_{n+1}$. Observe that

(1) by [**Hy**, Proposition 2.7], $M^K = \bigcup\{(M_n)^K | n = 1, 2, \dots\}$,

(2) by [**Hy**, Lemma 8.1.b], $(M_n)^K$ is a closed subspace of $(M_{n+1})^K$ for each n, and

(3) each $(M_n)^K$ is an ANR complete metric space (so, it is an ANE($\mathcal{C}$) by [**M**, Theorem 3.1(b)]).

Therefore, by [**Hy**, Theorem 11.3], $\operatorname{dirlim}(M_n)^K \in \mathcal{CW}(\mathcal{M})$ (or an M-space of [**Hy**]) and is an ANE($\mathcal{M}$). On the other hand, let P be a compact subset of M^K and $e\colon M^K \times K \to M$ the evaluation map. Then $e(P \times K)$ is a compact subset of M. So, there is an n such that $e(P \times K) \subset M_n$; in other words, $P \subset (M_n)^K$ for some n. Therefore, $M^K = \operatorname{dirlim}(M_n)^K$ by [**Hy**, Lemma 5.5]. Also, observe that M^K is an ANE($\mathcal{C}$) by use of (3).

Now, it follows from [**Hy**, Theorem 10.2] and its proof that $M^K = \operatorname{dirlim}(M_n)^K$ is an ANE($\mathcal{CW}(\mathcal{M})$) and an ANE($\mathcal{CW}(\mathcal{C})$). Finally, it follows easily that $C(K, M)$ is an ANE($\mathcal{CW}(\mathcal{M})$) and an ANE($\mathcal{CW}(\mathcal{C})$) by use of [**Gr**, 8.8] and $\mathcal{CW}(\mathcal{M}) \cup \mathcal{CW}(\mathcal{C}) \subset \mathcal{CG}$ [$\mathbf{L_3}$, §0]. □

LEMMA 2.2. *Let K be a finite complex and $N = K \times E^\infty$. Let A be a closed subset (G_δ-subset, resp.) of $B \in \mathcal{CW}(\mathcal{M})$ ($B \in \mathcal{CW}(\mathcal{C})$, resp.). Then each map $f\colon A \to \operatorname{Homeo}(N) \subset \operatorname{Homo}(N)$ has a continuous extension $g\colon U \to \operatorname{Homo}(N)$ over a neighborhood U of A in B; i.e., the associate $g\colon U \times N \to U \times N$ of g is an f.p. homotopy equivalence.*

PROOF. Identify K with $K \times 0 \subset N$ and let $\hat{f}\colon A \times N \to A \times N$ be the associate of f. We will first show that $\hat{f}$ has an f.p. extension $q\colon V \times N \to V \times N$, where V is a closed neighborhood of A in B. Then, we will choose a suitable neighborhood U of A in V such that $\hat{g} = q|U \times N$ is an f.p. homotopy equivalence.

Define

$$f_K\colon A \xrightarrow{f} \operatorname{Homeo}(N) \subset C(N, N) \xrightarrow{\tilde{i}_K} C(K, N),$$

where $\tilde{i}_K$ is induced from i_K. We have f_K being continuous. It follows from Lemma 2.1 that there is a closed neighborhood V of A in B and an extension $h\colon V \to C(K, N)$ of f_K. Let $\hat{h}\colon V \times K \to V \times N$ be its associate [**Gr**, 8.17] and $R\colon N \times I \to N$ a strong deformation retraction of N onto K. Define $d\colon V \times N \to V \times N$ by $d = \hat{h}(\mathrm{id}_V \times R_1)$. Then,

$$\begin{aligned} d|A \times N &= \hat{h}(\mathrm{id}_A \times R_1) = \hat{f}_K(\mathrm{id}_A \times R_1) = \hat{f}(\mathrm{id}_A \times R_1) \\ &\simeq \hat{f}(\mathrm{id}_A \times \mathrm{id}_N) \quad \text{(f.p.) since } R_1 \simeq \mathrm{id}_N \\ &= \hat{f}. \end{aligned}$$

Therefore, by [$\mathbf{L_3}$, Lemma 1.1], d is f.p. homotopic to a map $q\colon V \times N \to V \times N$ such that $q|A \times N = \hat{f}$.

Fix an $a \in A$. Consider $\hat{G} = q(\mathrm{id}_V \times \hat{f}_a^{-1})$ with its continuous associate $G\colon V \to C(N,N)$. We have $G(a) = \hat{f}_a \hat{f}_a^{-1} = \mathrm{id}_N$. From the proof of [$\mathbf{L}_4$, Theorem 1], there is an open neighborhood W of id_N in $C(N,N)$ such that for every map $s\colon X \to W$, its associate $\hat{s}\colon X \times N \to X \times N$ is f.p. homotopic to $\mathrm{id}_{X\times N}$. Let $U_a = G^{-1}(W)$. Then, the restriction $\hat{G}|U_a \times N$ is an f.p. homotopy equivalence; hence, so is $q|U_a \times N = \hat{G}(\mathrm{id}_{U_a} \times \hat{f}_a)$. Define $U = \bigcup\{U_a | a \in A\}$. Then, $\hat{g} = q|U \times N$ is a local f.p. homotopy equivalence; hence, it is an f.p. homotopy equivalence by [**Ja**, Theorem 5.57], and its associate $g\colon U \to \mathrm{Homo}(N)$ is an extension of f as desired. □

THEOREM 2.3. *If M is an E^∞-manifold having homotopy type of a finite complex, then* $\mathrm{Homeo}(M)$ *is an* $\mathrm{ANE}(\mathcal{CW}(\mathcal{M}))$ *and an* $\mathrm{ANE}(\mathcal{CW}(\mathcal{C}), G_\delta)$; *hence, so is* $k\,\mathrm{Homeo}(M)$.

PROOF. By [**He**, Theorem C], we can write $M = K \times E^\infty$, where K is a finite complex. Let A be a closed subset (G_δ-subset, resp.) of $B \in \mathcal{CW}(\mathcal{M})$ ($B \in \mathcal{CW}(\mathcal{C})$, resp.), and $f\colon A \to \mathrm{Homeo}(M)$ a map. Recall that since $A, B \in \mathcal{CG}$, we can use either topology on $\mathrm{Homeo}(M)$ and $\mathrm{Homo}(M)$. From Lemma 2.2, f has an extension $g\colon U \to \mathrm{Homo}(M)$, where U is a closed neighborhood of A in B, such that its associate $\hat{g}\colon U \times M \to U \times M$ is an f.p. homotopy equivalence. Then, it follows from Proposition 1.4 that there is an f.p. map $\hat{h}\colon U \times M \to U \times M$ whose associate $h\colon U \to C(M,M)$ satisfies the following properties:

(a) $h(U) \subset \mathrm{Homeo}(M)$, and

(b) $h|A = f$.

In other words, h is an extension of f over U into $\mathrm{Homeo}(M)$. □

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