SOME HOMOTOPY PROPERTIES OF THE HOMEOMORPHISM GROUPS OF $R^\infty(Q^\infty)$-MANIFOLDS

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ABSTRACT. In this note we will prove that, given an $R^\infty(Q^\infty)$-manifold $M$, there is a deformation of $\text{Homo}(M)$ into $\text{Homeo}(M)$ whose final stage is a weak homotopy equivalence, and that if $M$ has the homotopy type of a finite simplicial complex, then $\text{Homeo}(M)$ is an ANE($\mathcal{CW}(M)$) and an ANE($\mathcal{CW}(C), G_6$).

0. Introduction. All spaces are Hausdorff and maps are continuous. The composition of two maps $f: A \to B$ and $g: B \to C$ will be denoted by $gf$. Let $\mathcal{CG}$ denote the category whose objects are compactly generated spaces ($k$-spaces) and whose morphisms are continuous maps [Gr, 8.1]. Given two spaces $X$ and $Y$, following [Gr], let $X \times Y$ denote the product of $X$ and $Y$ in $\mathcal{CG}$, let $X \times_c Y$ be the Cartesian product, let $C(X, Y)$ be the space of maps from $X$ to $Y$ with the compact-open topology, and let $Y^X = kC(X, Y)$ [Gr, 8.14]. Similarly, let $\text{Homeo}(X)$ (Homo($X$), resp.) denote the space of homeomorphisms (homotopy equivalences, resp.) of $X$ with compact-open topology as a subspace of $C(X, X)$. Recall that $k\text{Homeo}(X)$ and $k\text{Homo}(X)$ are subspaces of $X^X$ in the sense of [Hy, p. 201]. Following [L3], let $\mathcal{CW}(C)$ and $\mathcal{CW}(M)$ denote the classes of pseudo $CW$-complexes generated by the class $\mathcal{C}$ of Hausdorff compact spaces and by the class $M$ of metric spaces, respectively. Let $E^\infty = \text{dirlimi} E^n$, where $E^n$ is either the $n$-Euclidean space or the $n$-fold product of the Hilbert cube $Q$. By an $E^\infty$-manifold, we mean a separable paracompact space which is locally homeomorphic to $E^\infty$.

In this note we will prove that given an $E^\infty$-manifold $M$, there is a deformation of $\text{Homo}(M)$ in itself into $\text{Homeo}(M)$ whose final stage is a weak homotopy equivalence (this partially answers a question raised by P. J. Kahn some time ago [L3, §0]), and that if $M$ has the homotopy type of a finite complex, then $\text{Homeo}(M)$ is an absolute neighborhood extensor [Hu] (abbreviated ANE) for the class $\mathcal{CW}(M)$ and is an ANE($\mathcal{CW}(C), G_6$). By a space $S \in \text{ANE}(\mathcal{C}, G_6)$, we mean that if $X \in \mathcal{C}$ and $A$ is a closed $G_6$-subset of $X$, then every map $f: A \to S$ has a continuous extension over a neighborhood of $A$ in $X$.

Throughout this note $I = [0, 1]$. By an $E^\infty$-manifold fiber bundle, we mean a locally trivial bundle in $\mathcal{CG}$, $p: X \to B$, whose fiber is an $E^\infty$-manifold. If $(X, p, B)$ is a bundle, $W$ a subset of $X$, and $A$ a subset of $B$, we write $W_A = W \cap p^{-1}(A)$; specially, $X_A = p^{-1}(A)$. Let $(X, p, B)$ and $(Y, p', B)$ be bundles and $W$ a subset of $X$. A map $f: W \to Y$ is said to be fiber preserving (f.p.) if $p'f = pW$. If $f: W \to B \times Z$ is an f.p. map, for each $b \in B$ let $f_b: W(b) \to Z$ denote the map $pZf|W(b)$ where $pZ: B \times Z \to Z$ is the projection. Given an $R^\infty(Q^\infty)$-manifold $M$, 

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following \([L_1, L_2]\), we write \(M = \text{dirlim} M_n\), where \(M_n\) is a compact submanifold (compact \(Q\)-submanifold) of \(M_{n+1}\) for each \(n = 1, 2, \ldots\).

Given a map \(f : B \to Y^X\) or \(f : B \to C(X, Y)\), by the associate of \(f\) we mean the f.p. function \(\hat{f} : B \times Y \to B \times Y\) defined by \(\hat{f}(b, x) = (b, f(b)(x))\), and vice versa; we also call \(f\) the associate of \(\hat{f}\) \([D, p. 261]\). By \([Gr, 8.17]\), the continuity of \(f\) and that of its associate are equivalent if \(B, X, Y \in \mathcal{C}\). Given a subset \(A\) of \(X\), let \(i_A\) denote the inclusion \(A \to X\) and \(id_A\) the identity map on \(A\).

1. A canonical deformation. The main result of this section is Theorem 1.3 which proves the existence of a deformation of the space of homotopy equivalences into the space of homeomorphisms. The proof of the following lemma is straightforward.

**Lemma 1.1.** Let \(X, Y,\) and \(Z\) be topological spaces. Let \(F : C(X, Y) \to C(X \times_c Z, Y \times_c Z)\) be the map defined by \(F(g) = g \times id_Z\). Then \(F\) is an embedding, i.e., \(C(X, Y)\) is homeomorphic to the subspace \(F(C(X, Y))\) of \(C(X \times_c Z, Y \times_c Z)\). \(\square\)

We need another lemma for the proof of Theorem 1.3. Recall that \(X \times_c I \cong X \times I\) for \(X \in \mathcal{C}\) \([Gr, 8.11]\).

**Lemma 1.2.** Let \(M, N\) be topological spaces. If \(g : M \times_c I \to M\) and \(h : N \times_c I \to N\) are given homotopies, then \(g\) and \(h\) induce the homotopies \(G\) and \(H\) on \(C(M, N)\) defined by \(G(f, t) = f g_t\) and \(H(f, t) = h_t f\) for all \(f \in C(M, N)\) and \(t \in I\).

**Proof.** For the continuity of \(G\), consider the following composition:

\[
\tilde{g} : C(M, N) \xrightarrow{\tilde{p}_M} C(M \times_c I, N) \xrightarrow{\tilde{g}'} C(M \times_c I, N) \xrightarrow{\sigma} C(I, C(M, N)),
\]

where \(\tilde{p}_M, \tilde{g}'\) are maps naturally induced \([D, p. 259]\) from the projections \(p_M : M \times_c I \to M\) and \(g' : M \times_c I \to M \times_c I\) defined by \(g'(x, t) = (g(x, t), t)\), and where \(\sigma\) is the embedding defined by \(\sigma(f)(t)(x) = f(x, t)\) for all \(t \in I\) and \(x \in M\) \([J, \text{Theorem 1.1}]\). Define

\[
G : C(M, N) \times_c I \xrightarrow{\bar{g} \times id_I} C(I, C(M, N)) \times_c I \xrightarrow{\sigma} C(M, N),
\]

where \(\bar{g}\) is the evaluation map. Then \(G\) is a homotopy. Let \(f \in C(M, N)\) and \(t \in I\). We have

\[
G(f, t) = e((\bar{g} \times id_I)(f, t)) = e((\sigma \tilde{g}' \tilde{p}_M)(f)) = e(\sigma(f p_M g')(t)) = f p_M g'(-, t) = f g_t.
\]

For the continuity of \(H\), consider the composition

\[
\tilde{h} : C(M, N) \xrightarrow{F} C(M \times_c I, N \times_c I) \xrightarrow{\tilde{h}} C(M \times_c I, N) \xrightarrow{\sigma} C(I, C(M, N)),
\]

where the embedding \(F\) is given by Lemma 1.1, \(\tilde{h}\) is induced from \(h\) and \(\sigma\) is from \([J, \text{Theorem 1.1}]\) as above. Define

\[
H : C(M, N) \times_c I \xrightarrow{\bar{h} \times id_I} C(I, C(M, N)) \times_c I \xrightarrow{\sigma} C(M, N).
\]
Then $H$ is continuous. Finally, let $f \in C(M, N)$ and $t \in I$. We have

$$H(f, t) = e(\tilde{h} \times \text{id}_I)(f, t) = e(\tilde{h}(f), t) = \tilde{h}(f)(t)$$

$$= (\sigma \tilde{h}F(f))(t) = \tilde{h}(F(f))(\cdot, t)$$

$$= (\tilde{h}(f \times \text{id}_I))(\cdot, t) = h((f \times \text{id}_I)(\cdot, t))$$

$$= h(f(\cdot), t) = htf. \quad \square$$

**Remark 1.** In Lemma 1.2, if $M$ and $N$ are $k$-spaces, then $G$ and $H$ are also homotopies on $N^M = kC(M, N)$ (by use of [Gr, 8.8(vi)]).

**Theorem 1.3.** Let $M$ be an $E^\infty$-manifold. Then, there is a deformation of $\text{Homo}(M)$ in itself into $\text{Homeo}(M)$ whose final map $r: \text{Homo}(M) \rightarrow \text{Homeo}(M)$ is a weak homotopy equivalence.

**Proof.** To avoid confusion, we let $N$ be another copy of $M$ and consider $\text{Homo}(M, N)$ and $\text{Homeo}(M, N)$ instead of $\text{Homo}(M)$ and $\text{Homeo}(M)$, respectively. By [He, Theorem C], we can identify $M$ and $N$ with $K \times E^\infty$ and $L \times E^\infty$, respectively, where $K$ and $L$ are locally finite simplicial complexes. We also identify $K$ and $L$ with $K \times 0$ and $L \times 0$, respectively. Let $g: M \times I \rightarrow M$ and $h: N \times I \rightarrow N$ be natural strong deformation retractions of $M$ and $N$ onto $K$ and $L$, respectively, with $g_1 = p_K$ and $h_1 = p_L$ (the projections). Let $G$ and $H$ be the homotopies on $C(M, N)$ induced by $g$ and $h$ given by Lemma 1.2. Define $D_t: C(M, N) \rightarrow C(M, N)$ by

$$D_t = \begin{cases} H_{4t} & \text{if } t \in [0, 1/4], \\ G_{4t-1} & \text{if } t \in [1/4, 1/2], \\ G_{4t-2} & \text{if } t \in [1/2, 3/4], \\ H_{4t-3} & \text{if } t \in [3/4, 1], \end{cases}$$

where $G_t^* = G_{1-t}$ and $H_t^* = H_{1-t}$. Then

$$D: f \simeq p_Lf \simeq p_Lf p_K \simeq p_L((p_Lf)(K) \times \text{id}_{E^\infty}) \simeq (p_Lf)(K) \times \text{id}_{E^\infty}, w3$$

and

$$D_t(C(M, N)) = \{q \times \text{id}_{E^\infty} | q \in C(K, L)\} \cong C(K, L)$$

(by Lemma 1.1) which is a metrizable space [Ku, Theorem 1, p. 93].

Observe that $D(\text{Homo}(M, N) \times_{cI}) \subset \text{Homo}(M, N)$. So, $D$ deforms $\text{Homo}(M, N)$ in itself into the metrizable subspace $B \equiv \text{Homo}(M, N) \cap \{q \times \text{id}_{E^\infty} | q \in C(K, L)\}$. Recall that since $B$ is metrizable, it also is a subspace of $k\text{Homo}(M, N)$. Now, we would like to deform $B$ into $\text{Homeo}(M, N)$. Consider the associate $g$ of the inclusion $i_B: B \rightarrow \text{Homo}(M, N)$, i.e.,

$$g: B \times K \times E^\infty \rightarrow B \times L \times E^\infty$$

defined by $g(b, x, y) = (b, q(x), y)$, where $b = q \times \text{id}_{E^\infty}$. Then, $g$ is continuous [Gr, 8.17], and $g$ is an f.p. homotopy equivalence [Ja, Proposition 7.58]. Therefore, by [L3, Lemma 3.3], $g$ is f.p. homotopic to an f.p. homeomorphism, say $\tilde{g}$. This homotopy induces a homotopy $R: B \times I \rightarrow \text{Homo}(M, N)$ with $R_0 = i_B$ and $R_1(B) \subset \text{Homeo}(M, N)$. Combining $D$ and $R$, we will obtain a desired deformation with a final map $r = R_1D_1$.

Finally, let $i: \text{Homeo}(M) \rightarrow \text{Homo}(M)$ be the natural inclusion. Then we have

$$ir = r \simeq \text{id}_{\text{Homo}(M)}.$$
induces isomorphisms on homotopy groups). It follows easily that $r$ is also a weak homotopy equivalence.

To conclude this section, we prove the following proposition which will be used in the next section. In fact, the proposition indicates that $\text{Homeo}(M)$ is not a closed subset of $\text{Homo}(M)$. Therefore, it is interesting to know whether the deformation in the above theorem can be chosen such that $\text{Homeo}(M)$ is deformed in itself.

**PROPOSITION 1.4.** Let $A$ be a closed subset ($G_\delta$-subset, resp.) of $B \in CW(M)$ ($B \in CW(C)$, resp.), and $(X,p,B)$ an $E^\infty$-manifold fiber bundle. If $f: X \to X$ is an f.p. homotopy equivalence with $f|X_A$ a continuous injection, then there is an f.p. map $h: X \to X$ such that $h|X_{B-A}: X_{B-A} \to X_{B-A}$ is an f.p. homeomorphism and that $f$ is f.p. homotopic to $h$ (rel $X_A$).

**PROOF.** Similar to the proof of [L3, Theorem 3.4], we will assume that $X = B \times M$, where $M$ is an $E^\infty$-manifold. Since $A$ is a closed $G_\delta$-set in $B$, we can write $B - A = \bigcup \{C_n|n = 1, 2, \ldots\}$, where each $C_n$ is a closed $G_\delta$-set in $B$ and $C_n \subset$ (interior of $C_{n+1}$). Write $M = \operatorname{dirlim} M_n$ and assume $C_0 = \emptyset$ and $M_0 = 0$. We now define by induction a sequence of f.p. homotopies $H^n: B \times M \times I \to B \times M$ and $h_n = H^n_1$ such that

(i) $H^n_1: h_{n-1} \simeq h_n$ (f.p.) rel $(A \times M) \cup (C_{n-1} \times M) \cup (B \times M_{n-1})$,

(ii) $h_n|C_1 \times M: C_1 \times M \to C_1 \times M$ is an f.p. homeomorphism, and

(iii) $h_n|B \times M_n: B \times M_n \to B \times M$ is an f.p. embedding.

First, since $f|A \times M_1$ is an f.p. embedding, there is from [L3, Lemma 3.3] an f.p. homotopy $F: B \times M \times I \to B \times M$ (rel $A \times M_1$) such that $F_0 = f$ and $F_1$ is an f.p. homeomorphism. Let $s: B \to I$ be a map with $s^{-1}(0) = A$ and $s^{-1}(1) = C_1$. Define $H^1: B \times M \times I \to B \times M$ by $H^1(b, x, t) = F(b, x, s(b)t)$. Then we have

(i) $H^1_1: f (= h_0) \simeq h_1$ (f.p.) rel $A \times M$,

(ii) $h_1|C_1 \times M = F_1|C_1 \times M$ is an f.p. homeomorphism, and

(iii) $h_1|B \times M_1: B \times M_1 \to B \times M$ is an f.p. embedding, by use of [L3, Lemma 1.2].

Second, assume that $H^{n-1}$ has been defined such that (i)$_{n-1}$, (ii)$_{n-1}$, and (iii)$_{n-1}$ are satisfied. By [L3, Lemma 3.3], there is an f.p. homotopy $G: B \times M \times I \to B \times M$ (rel $A \times M_n \cup C_{n-1} \times M \cup B \times M_{n-1}$) such that $G_0 = h_{n-1}$ and $G_1$ is an f.p. homeomorphism. Similar to above, by use of an Urysohn function and [L3, Lemma 1.2], we can obtain a homotopy $H^n$ such that (i)$_n$, (ii)$_n$, and (iii)$_n$ are satisfied.

Finally, define $h = \lim h_n$; then, $h$ is well defined and continuous by use of (iii)$_n$; and, $h|(B - A) \times M$ is an f.p. homeomorphism by use of (ii)$_n$ and the condition $C_n \subset$ (interior of $C_{n+1}$). Moreover, a desired homotopy $H: B \times M \times I \to B \times M$ from $f$ to $h$ can be defined by

$$H(b, x, t) = \begin{cases} H^n(b, x, 2^n(t - 1) + 2) & \text{if } t \in J_n, \\ h(x) & \text{if } t = 1, \end{cases}$$

where $J_n = [1 - (1/2^n-1), 1 - (1/2^n)]$. Then, $H$ is well defined and continuous by (i)$_n$.

2. Absolute neighborhood extensor properties of $\text{Homeo}(M)$. We first prove some lemmas that will be used in the proof of the main result of this section, Theorem 2.3.
LEMMA 2.1. Let $K$ be a compact metric space and $M$ an $E^\infty$-manifold. Then the space $M^K \in \mathcal{CW}(M)$; moreover, it is an ANE($\mathcal{M}$) and an ANE($\mathcal{C}$). Consequently, $M^K$ is an ANE($\mathcal{CW}(M)$) and an ANE($\mathcal{CW}(C)$), and so is $C(K,M)$.

PROOF. The proof is implicit in that of [Hy, Theorem 8.2]. Because of its simplicity in this case, we will give its outline.

Write $M = \varinjlim M_n$, where each $M_n$ is a compact ANR metric space with $M_n \subset M_{n+1}$. Observe that

1. by [Hy, Proposition 2.7], $M^K = \bigcup \{(M_n)^K | n = 1, 2, \ldots \}$,
2. by [Hy, Lemma 8.1.b], $(M_n)^K$ is a closed subspace of $(M_{n+1})^K$ for each $n$, and
3. each $(M_n)^K$ is an ANR complete metric space (so, it is an ANE($\mathcal{C}$) by [M, Theorem 3.1(b)]).

Therefore, by [Hy, Theorem 11.3], $\varinjlim (M_n)^K \in \mathcal{CW}(M)$ (or an $M$-space of $[Hy]$) and is an ANE($\mathcal{M}$). On the other hand, let $P$ be a compact subset of $M^K$ and $e: M^K \times K \to M$ the evaluation map. Then $e(P \times K)$ is a compact subset of $M$. So, there is an $n$ such that $e(P \times K) \subset M_n$; in other words, $P \subset (M_n)^K$ for some $n$. Therefore, $M^K = \varinjlim (M_n)^K$ by [Hy, Lemma 5.5]. Also, observe that $M^K$ is an ANE($\mathcal{C}$) by use of (3).

Now, it follows from [Hy, Theorem 10.2] and its proof that $M^K = \varinjlim (M_n)^K$ is an ANE($\mathcal{CW}(M)$) and an ANE($\mathcal{CW}(C)$). Finally, it follows easily that $C(K,M)$ is an ANE($\mathcal{CW}(M)$) and an ANE($\mathcal{CW}(C)$) by use of [Gr, 8.8] and $\mathcal{CW}(M) \cup \mathcal{CW}(C) \subset \mathcal{Cg}$ [L3, §0]. □

LEMMA 2.2. Let $K$ be a finite complex and $N = K \times E^\infty$. Let $A$ be a closed subset (G$_s$-subset, resp.) of $B \in \mathcal{CW}(M)$ ($B \in \mathcal{CW}(C)$, resp.). Then each map $f: A \to \text{Homeo}(N) \subset \text{Homo}(N)$ has a continuous extension $g: U \to \text{Homo}(N)$ over a neighborhood $U$ of $A$ in $B$; i.e., the associate $g: U \times N \to U \times N$ of $g$ is an f.p. homotopy equivalence.

PROOF. Identify $K$ with $K \times 0 \subset N$ and let $\hat{f}: A \times N \to A \times N$ be the associate of $f$. We will first show that $\hat{f}$ has an f.p. extension $q: V \times N \to V \times N$, where $V$ is a closed neighborhood of $A$ in $B$. Then, we will choose a suitable neighborhood $U$ of $A$ in $V$ such that $\hat{g} = q|U \times N$ is an f.p. homotopy equivalence.

Define

\[ f_K: A \xrightarrow{f} \text{Homeo}(N) \subset C(N,N) \xrightarrow{i_K} C(K,N), \]

where $i_K$ is induced from $i_K$. We have $f_K$ being continuous. It follows from Lemma 2.1 that there is a closed neighborhood $V$ of $A$ in $B$ and an extension $h: V \to C(K,N)$ of $f_K$. Let $\tilde{h}: V \times K \to V \times N$ be its associate [Gr, 8.17] and $R: N \times I \to N$ a strong deformation retraction of $N$ onto $K$. Define $d: V \times N \to V \times N$ by $d = \tilde{h}(\text{id}_V \times R_1)$. Then,

\[ d|A \times N = \tilde{h}(\text{id}_A \times R_1) = f_K(\text{id}_A \times R_1) = f(\text{id}_A \times R_1) \]
\[ \simeq f(\text{id}_A \times \text{id}_N) \quad \text{(f.p.)} \quad \text{since } R_1 \simeq \text{id}_N \]
\[ = \hat{f}. \]

Therefore, by [L3, Lemma 1.1], $d$ is f.p. homotopic to a map $q: V \times N \to V \times N$ such that $q|A \times N = \hat{f}$. 

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Fix an \( a \in A \). Consider \( \tilde{G} = q(id_N \times \tilde{f}_a^{-1}) \) with its continuous associate \( G: V \to C(N, N) \). We have \( G(a) = \tilde{f}_a \tilde{f}_a^{-1} = id_N \). From the proof of [L4, Theorem 1], there is an open neighborhood \( W \) of \( id_N \) in \( C(N, N) \) such that for every map \( s: X \to W \), its associate \( \tilde{s}: X \times N \to X \times N \) is f.p. homotopic to \( id_{X \times N} \). Let \( U_a = G^{-1}(W) \). Then, the restriction \( G|U_a \times N \) is an f.p. homotopy equivalence; hence, so is \( q|U_a \times N = \tilde{G}(id_{U_a} \times \tilde{f}_a) \). Define \( U = \bigcup \{ U_a | a \in A \} \). Then, \( \hat{g} = q|U \times N \) is a local f.p. homotopy equivalence; hence, it is an f.p. homotopy equivalence by [Ja, Theorem 5.57], and its associate \( g: U \to \text{Homo}(N) \) is an extension of \( f \) as desired. \( \square \)

**Theorem 2.3.** If \( M \) is an \( E^\infty \)-manifold having homotopy type of a finite complex, then \( \text{Homeo}(M) \) is an ANE(\( CW(M) \)) and an ANE(\( CW(C), G_\delta \)); hence, so is \( k\text{Homeo}(M) \).

**Proof.** By [He, Theorem C], we can write \( M = K \times E^\infty \), where \( K \) is a finite complex. Let \( A \) be a closed subset \( (G_\delta \)-subset, resp.) of \( B \in CW(M) \) \( (B \in CW(C), \text{resp.}) \), and \( f: A \to \text{Homeo}(M) \) a map. Recall that since \( A, B \in CW \), we can use either topology on \( \text{Homeo}(M) \) and \( \text{Homo}(M) \). From Lemma 2.2, \( f \) has an extension \( g: U \to \text{Homo}(M) \), where \( U \) is a closed neighborhood of \( A \) in \( B \), such that its associate \( \hat{g}: U \times M \to U \times M \) is an f.p. homotopy equivalence. Then, it follows from Proposition 1.4 that there is an f.p. map \( h: U \times M \to U \times M \) whose associate \( h: U \to \text{C}(M, M) \) satisfies the following properties:

\begin{enumerate}
  \item \( h(U) \subset \text{Homeo}(M) \), and
  \item \( h|A = f \).
\end{enumerate}

In other words, \( h \) is an extension of \( f \) over \( U \) into \( \text{Homeo}(M) \). \( \square \)

**References**


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