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SHORTER NOTES

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EMBEDDING AND UNKNOTTING OF SOME POLYHEDRA

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ABSTRACT. If a compact polyhedron X^n, $n \geq 3$ (resp. $n \geq 2$), has the property that any two of its nonsingular points can be joined by an arc containing at most one singular point, then X^n embeds in $\mathbf{R}^{2n}$ (resp. unknots in $\mathbf{R}^{2n+1}$).

The object of this note is to discuss a situation where the Penrose-Whitehead-Zeeman construction (see Zeeman [**10**, pp. 66–67]) works for a class of polyhedra much more general than manifolds. In particular reduced polyhedra satisfy our hypotheses. Thus Husch's unknotting theorem [**4**] is a special case of the result proved below.

Let X be a compact polyhedron of dimension n. A point x of X is called *nonsingular* (resp. *singular*) if there exists (resp. does not exist) a triangulation of X containing x in the interior of an n-simplex.

THEOREM. *Let X be a compact polyhedron of dimension $n \geq 3$ (resp. $n \geq 2$). If any two nonsingular points of X can be joined by an arc containing at most one singular point, then X embeds in $\mathbf{R}^{2n}$ (resp. unknots in $\mathbf{R}^{2n+1}$).*

Embedding. General position yields a p.l. map $f\colon X^n \to \mathbf{R}^{2n}$ with a finite number of nonsingular double points. To explain our iterative construction it suffices to consider the case when there is just one pair $\{x_1, x_2\}$ of nonsingular double points, $f(x_1) = f(x_2)$. Let A be an arc, containing at most one singular point of X, and joining x_1 to x_2. Because $2 + n < 2n$ any general position point p of $\mathbf{R}^{2n}$ is joinable to the circle $C = f(A)$ in such a way that the 2-disk $D = pC$ meets $f(X^n)$ in precisely C. By choosing triangulations of X^n (*resp.* $\mathbf{R}^{2n}$) in which A (*resp.* $f(X^n)$ and D) are full subcomplexes, and f is simplicial, we can find regular neighborhoods $N(A)$ of A in X, and $N(D)$ of D in $\mathbf{R}^{2n}$, such that $f(X - N(A)) \subseteq \mathbf{R}^{2n} - N(D)$, $f(\partial N(A)) \subseteq \partial N(D)$ and $f(N(A)) \subseteq N(D)$. If A has no singular point, $N(A)$ is an n-disk. If A has the unique singular point y, then $N(A)$ is p.l. homeomorphic to the closed star of y. In either case we see that $N(A)$ is a cone over its boundary $\partial N(A)$. Therefore we can extend the embedding

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$f \mid (X^n - \mathrm{int}N(A))$ to an embedding $g\colon X^n \to \mathbf{R}^{2n}$ by coning $f(\partial N(A))$ over an interior point of the $2n$-disk $N(D)$.

Unknotting. General position yields a p.l. map $f\colon X^n \times I \to \mathbf{R}^{2n+1} \times I$ whose 'ends' f_0, f_1 are two given embeddings of X^n in $\mathbf{R}^{2n+1}$, and which has a finite number of nonsingular double points. Since any two nonsingular points of $X^n \times I$ too can be joined by an arc having at most one singular point, we repeat the above construction to get a p.l. embedding $g\colon X^n \times I \to \mathbf{R}^{2n+1} \times I$ with ends $g_0 = f_0$, $g_1 = f_1$. Thus f_0 and f_1 are concordant. By Lickorish [**5**, Theorem 6], concordance implies isotopy in codimensions ≥ 3. Thus f_0 and f_1 are isotopic.

The above theorem is best possible in the sense that one cannot replace 'at most one' by 'at most two'. Recall that two n-spheres can link in $\mathbf{R}^{2n+1}$. Thus, by joining two n-spheres, $n \geq 1$, by a thin 'ribbon', we get an example of an n-dimensional polyhedron which knots in $\mathbf{R}^{2n+1}$ and for which any two nonsingular points can be joined by an arc containing at most two singular points. Another example of a polyhedron having this joinability property is the n-skeleton of an N-simplex, $N \geq 2n+1$, $n \geq 1$. It was proved by van Kampen [**7**] and Flores [**3**] that, for $N \geq 2n+2$, this polyhedron does not embed in $\mathbf{R}^{2n}$.

Husch unknotting. A homogenously n-dimensional and connected polyhedron X^n is called *reduced* if it can be obtained from some other, Y^n, by replacing a regular neighborhood $N(T)$, of a maximal tree T of a triangulation of Y^n, by a cone $z \cdot \partial N(T)$. Since T is a maximal tree, for each $x \in Y$ we can find a $t \in T$ and an arc α from x to t such that all points of $\alpha - \{x, t\}$ are nonsingular points of Y^n. From this it follows that any nonsingular point of X^n can be joined to $\partial N(T)$ via nonsingular points of X^n, and thus, that any two nonsingular points of X can be joined by an arc A through the base point z, such that all points of $A - \{z\}$ are nonsingular points of X^n. Therefore the above theorem implies Husch's result [**4**] that all reduced polyhedra X^n, $n \geq 2$, unknot in $\mathbf{R}^{2n+1}$. Note that the n-skeleton of a $2n$-simplex, $n \geq 2$, is not reduced, but does satisfy the hypothesis of the above theorem.

Bibliographical remarks. The case X^n = a connected pseudomanifold (resp. X^n = polyhedron obtained by making some identifications on the boundary of a connected manifold) of the above theorem is due to van Kampen [**7**] (resp. Edwards [**2**]). The construction given in the above proof (resp. general Penrose-Whitehead-Zeeman construction) is a variation (resp. a generalization) of a construction by which van Kampen [**7**] eliminates those pairs of double points, of a g.p. map $f\colon |K^n| \to \mathbf{R}^{2n}$, which lie in adjacent n-simplices of K^n. For other developments of van Kampen's ideas see also Shapiro [**6**], Wu [**9**] and Weber [**8**]. For more on singularities see Akin [**1**].

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