EMBEDDING AND UNKNOTTING OF SOME POLYHEDRA
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ABSTRACT. If a compact polyhedron $X^n$, $n \geq 3$ (resp. $n \geq 2$), has the property that any two of its nonsingular points can be joined by an arc containing at most one singular point, then $X^n$ embeds in $\mathbb{R}^{2n}$ (resp. unknots in $\mathbb{R}^{2n+1}$).

The object of this note is to discuss a situation where the Penrose-Whitehead-Zeeman construction (see Zeeman [10, pp. 66-67]) works for a class of polyhedra much more general than manifolds. In particular reduced polyhedra satisfy our hypotheses. Thus Husch's unknotting theorem [4] is a special case of the result proved below.

Let $X$ be a compact polyhedron of dimension $n$. A point $x$ of $X$ is called nonsingular (resp. singular) if there exists (resp. does not exist) a triangulation of $X$ containing $x$ in the interior of an $n$-simplex.

THEOREM. Let $X$ be a compact polyhedron of dimension $n \geq 3$ (resp. $n \geq 2$). If any two nonsingular points of $X$ can be joined by an arc containing at most one singular point, then $X$ embeds in $\mathbb{R}^{2n}$ (resp. unknots in $\mathbb{R}^{2n+1}$).

Embedding. General position yields a p.l. map $f: X^n \to \mathbb{R}^{2n}$ with a finite number of nonsingular double points. To explain our iterative construction it suffices to consider the case when there is just one pair $\{x_1, x_2\}$ of nonsingular double points, $f(x_1) = f(x_2)$. Let $A$ be an arc, containing at most one singular point of $X$, and joining $x_1$ to $x_2$. Because $2 + n < 2n$ any general position point $p$ of $\mathbb{R}^{2n}$ is joinable to the circle $C = f(A)$ in such a way that the 2-disk $D = pC$ meets $f(X^n)$ in precisely $C$. By choosing triangulations of $X^n$ (resp. $\mathbb{R}^{2n}$) in which $A$ (resp. $f(X^n)$ and $D$) are full subcomplexes, and $f$ is simplicial, we can find regular neighborhoods $N(A)$ of $A$ in $X$, and $N(D)$ of $D$ in $\mathbb{R}^{2n}$, such that $f(X - N(A)) \subseteq \mathbb{R}^{2n} - N(D)$, $f(\partial N(A)) \subseteq \partial N(D)$ and $f(N(A)) \subseteq N(D)$. If $A$ has no singular point, $N(A)$ is an $n$-disk. If $A$ has the unique singular point $y$, then $N(A)$ is p.l. homeomorphic to the closed star of $y$. In either case we see that $N(A)$ is a cone over its boundary $\partial N(A)$. Therefore we can extend the embedding...
Unknotting. General position yields a p.l. map \( f: X^n \times I \to \mathbb{R}^{2n+1} \times I \) whose 'ends' \( f_0, f_1 \) are two given embeddings of \( X^n \) in \( \mathbb{R}^{2n+1} \), and which has a finite number of nonsingular double points. Since any two nonsingular points of \( X^n \times I \) too can be joined by an arc having at most one singular point, we repeat the above construction to get a p.l. embedding \( g: X^n \times I \to \mathbb{R}^{2n+1} \times I \) with ends \( g_0 = f_0, g_1 = f_1 \). Thus \( f_0 \) and \( f_1 \) are concordant. By Lickorish [5, Theorem 6], concordance implies isotopy in codimensions \( \geq 3 \). Thus \( f_0 \) and \( f_1 \) are isotopic.

The above theorem is best possible in the sense that one cannot replace 'at most one' by 'at most two'. Recall that two \( n \)-spheres can link in \( \mathbb{R}^{2n+1} \). Thus, by joining two \( n \)-spheres, \( n \geq 1 \), by a thin 'ribbon', we get an example of an \( n \)-dimensional polyhedron which knots in \( \mathbb{R}^{2n+1} \) and for which any two nonsingular points can be joined by an arc containing at most two singular points. Another example of a polyhedron having this joinability property is the \( n \)-skeleton of an \( N \)-simplex, \( N \geq 2n + 1, n \geq 1 \). It was proved by van Kampen [7] and Flores [3] that, for \( N \geq 2n + 2 \), this polyhedron does not embed in \( \mathbb{R}^{2n} \).

Husch unknotting. A homogenously \( n \)-dimensional and connected polyhedron \( X^n \) is called reduced if it can be obtained from some other, \( Y^n \), by replacing a regular neighborhood \( N(T) \), of a maximal tree \( T \) of a triangulation of \( Y^n \), by a cone \( z \cdot \partial N(T) \). Since \( T \) is a maximal tree, for each \( x \in Y \) we can find a \( t \in T \) and an arc \( \alpha \) from \( x \) to \( t \) such that all points of \( \alpha - \{x, t\} \) are nonsingular points of \( Y^n \). From this it follows that any nonsingular point of \( X^n \) can be joined to \( \partial N(T) \) via nonsingular points of \( X^n \), and thus, that any two nonsingular points of \( X \) can be joined by an arc \( A \) through the base point \( z \), such that all points of \( A - \{z\} \) are nonsingular points of \( X^n \). Therefore the above theorem implies Husch's result [4] that all reduced polyhedra \( X^n, n \geq 2 \), unknot in \( \mathbb{R}^{2n+1} \). Note that the \( n \)-skeleton of a \( 2n \)-simplex, \( n \geq 2 \), is not reduced, but does satisfy the hypothesis of the above theorem.

Bibliographical remarks. The case \( X^n = \) a connected pseudomanifold (resp. \( X^n = \) polyhedron obtained by making some identifications on the boundary of a connected manifold) of the above theorem is due to van Kampen [7] (resp. Edwards [2]). The construction given in the above proof (resp. general Penrose-Whitehead-Zeeman construction) is a variation (resp. a generalization) of a construction by which van Kampen [7] eliminates those pairs of double points, of a g.p. map \( f: |K^n| \to \mathbb{R}^{2n} \), which lie in adjacent \( n \)-simplices of \( K^n \). For other developments of van Kampen's ideas see also Shapiro [6], Wu [9] and Weber [8]. For more on singularities see Akin [1].

References

3. A. Flores, Über \( n \)-dimensionale Komplexe die im \( \mathbb{R}^{2n+1} \) absolut selbstverschlungen sind, Ergeb. Math. Kolloq. 6 (1933/34), 4–7.

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