A NOTE ON COCYCLES  
IN VON NEUMANN ALGEBRAS  
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ABSTRACT. In this note, we study the comparison theory for cocycles in von Neumann algebras. In particular, we investigate quasi-equivalent cocycles.

In [1], Connes and Takesaki studied a comparison theory for cocycles with respect to a given continuous group action on a von Neumann algebra. This theory will give rise, via the Connes cocycle theorem [1, 3.1, 3.5], to a corresponding comparison theory for weights on von Neumann algebras. Further, Muhly and the author in [2] proved that when a von Neumann algebra $M$ is in standard form, there is essentially a one-to-one correspondence between invariant subspaces of an analytic subalgebra of $M$ which is determined by an action $\{\alpha_t\}_{t \in \mathbb{R}}$ and cocycles for $\{\alpha_t\}_{t \in \mathbb{R}}$.

In this note, we shall develop a comparison theory for cocycles in a von Neumann algebra, in particular, a finite von Neumann algebra, and apply it to a comparison theory for invariant subspaces in von Neumann algebras.

Let $M$ be a von Neumann algebra and let $G$ be a locally compact group. Let $\alpha: G \to \text{Aut}(M)$ be a continuous action of $G$ on $M$. As in [4, 20.1], recall that a cocycle is an $s^*$-continuous function $\alpha: G \ni s \to a(s) \in M$ with the properties:

$$a(st) = a(s)\alpha_s(a(t)) \quad \text{and} \quad a(s^{-1}) = \alpha^{-1}_s(a(s)^*) , \quad s, t \in G,$$

and that the set of all cocycles is denoted by $Z_a(G, M)$. If $a \in Z_a(G, M)$, then the elements $a(s) \in M$ are partial isometries:

$$a(s)a(s)^* = a(1), \quad a(s)^*a(s) = \alpha_s(a(1)), \quad s \in G;$$

in particular, $a(1)$ is a projection, where $1$ means the identity of $G$. For each $s \in G$, we set

$$a\alpha_s(x) = a(s)\alpha_s(x)a(s)^*, \quad x \in Ma(1), \quad s \in G.$$

Then $a\alpha: G \to \text{Aut}(Ma(1))$ is a continuous action whose centralizer is denoted by $M^a = (Ma(1))^a$. If $p \in \text{Proj}(M^a)$, then the map $s \in G \to pa(s) \in M$ is a cocycle in $M$. We call it a subcocycle of $a$ and denote it by $a^p$.

Let $F_2$ be the type $I_2$-factor with the system of matrix units $\{e_{ij}\}_{1 \leq i, j \leq 2}$. We shall identify $M \otimes F_2$ with $\text{Mat}_2(M)$ in the usual way. Let $\iota: G \to \text{Aut}(F_2)$ be the trivial action. Then $\alpha \otimes \iota: G \to \text{Aut}(M \otimes F_2)$ is a continuous action. Given $a, b \in Z_a(G, M)$, we define the balanced cocycle $c = c(a, b)$ associated with the cocycles $a$ and $b$ by

$$c(s) = a(s) \otimes e_{11} + b(s) \otimes e_{22} = \begin{pmatrix} a(s) & 0 \\ 0 & b(s) \end{pmatrix} , \quad s \in G,$$
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and the set \( I(a, b) \) by

\[ I(a, b) = \{ x \in a(1)M_b(1) : xb(s) = a(s)a_s(x) \text{ for all } s \in G \} \]

**Definition 1.** We say that \( a, b \in \mathcal{Z}_\alpha(G, M) \) are equivalent and write \( a \simeq b \) if there exists an element \( c \in M \) such that

\[ a(s) = c^*b(s)a_s(c), \quad b(s) = ca(s)a_s(c^*), \quad s \in G. \]

We write \( a \preceq b \) if \( a \simeq b^p \) for some \( p \in \text{Proj}(M^b) \). Further, we say that \( a \) and \( b \) are disjoint and write \( a \perp b \) if \( a(1) \otimes e_{11} \) and \( b(1) \otimes e_{22} \) are centrally orthogonal in \( M^b_2 \). We say also that \( a \) and \( b \) are quasi-equivalent and write \( a \sim b \) if \( a \simeq b^p \) for some \( p \in \text{Proj}(M^b) \).

Let \( a, b \in \mathcal{Z}_\alpha(G, M) \). If \( a \perp b \), then \( I(a, b) = \{ 0 \} \) (cf. [4, Proposition 20.2]). We now study the structure of \( I(a, b) \) when \( a \) and \( b \) are not disjoint. Assume that \( I(a, b) \neq \{ 0 \} \). Let \( x \in I(a, b), \ x \neq 0 \). If \( x = \|x\| \) is the polar decomposition of \( x \), then \( |x| \in I(b, b) = M^b \) and \( v \in I(a, b) \). Thus we have \( v^*v \leq b(1) \) and \( vv^* \leq a(1) \).

We put

\[ p(a, b) = \sup\{ v^*v : v \text{ is a partial isometry in } I(a, b) \} \]

and

\[ q(a, b) = \sup\{ vv^* : v \text{ is a partial isometry in } I(a, b) \}, \]

respectively. Hence it is clear that the \( \sigma \)-weakly closed linear span \( J \) of \( I(a, b)^*I(a, b) \) is a \( \sigma \)-weakly closed 2-sided ideal of \( M^b \). Thus there exists a central projection \( e_0 \) in \( M^b \) such that \( J = M^b e_0 \). By the definition of \( p(a, b) \), we easily have \( p(a, b) = e_0 \) and so \( p(a, b) \) is a central projection in \( M^b \). Furthermore, \( p(a, b) \) is the least central projection in \( M^b \) such that \( xp(a, b)x = x \) \( (x \in I(a, b)) \). Similarly \( q(a, b) \) is the least central projection in \( M^a \) such that \( q(a, b)x = x \) for all \( x \in I(a, b)^* = I(b, a) \). For simplicity, we put \( p(a, b) = p \) and \( q(a, b) = q \), respectively.

**Proposition 2.** Let \( a, b \in \mathcal{Z}_\alpha(G, M) \). Then \( a^{a(1) - q} \perp b \) and \( a^q \sim b^p \).

**Proof.** Put \( r = a(1) - q \). Then it is sufficient to prove that \( I(a^r, b) = \{ 0 \} \).

Since \( r \in M^a \), \( a(t)\alpha_t(r)a(t)^* = r \). Let \( x \in I(a^r, b) \). Then

\[ xb(t) = ra(t)\alpha_t(x) = a(t)\alpha_t(r)a(t)^*a(t)\alpha_t(x) = a(t)\alpha_t(r)x = a(t)\alpha_t(r)ax = x. \]

Thus \( I(a^r, b) \subset I(a, b) \cap rM_b(1) = \{ 0 \} \). This implies that \( a^r \perp b \).

Next we shall prove that \( a^q \sim b^p \). To prove this, it is sufficient to prove that \( q \otimes e_{11} \) and \( p \otimes e_{22} \) have the same central support in \( M^b_2 \). Let \( \begin{pmatrix} z & y \\ z & w \end{pmatrix} \) be the central support of \( q \otimes e_{11} = \begin{pmatrix} q \otimes 0 \\ 0 \otimes q \end{pmatrix} \) in \( M^b_2 \). By [4, Proposition 20.2], it is clear that \( x \) (resp. \( y \)) is a central projection in \( M^a \) (resp. \( M^b \)), \( x = z = 0 \), and \( xd = dw \) for all \( d \in I(a, b) \).

On the other hand,

\[ \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \leq \begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} \leq \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \]

and so \( q = x \) and \( w \leq p \). Since \( d = qd = dw \) for all \( d \in I(a, b) \) and \( p \) is the least central projection in \( M^b \) such that \( dp = d \) for all \( d \in I(a, b) \), \( p = w \). Thus the central support of \( q \otimes e_{11} = q \otimes e_{11} + p \otimes e_{22} \). Similarly, we have the central support of \( p \otimes e_{22} = q \otimes e_{11} + p \otimes e_{22} \). Thus \( a^q \sim b^p \). This completes the proof.
If $M$ is $\sigma$-finite and $a, b \in Z_{\alpha}(G, M)$ are of infinite multiplicity, then $a \sim b$ is equivalent to $a \sim b$ [4, 20.2]. However, if $a, b \in Z_{\alpha}(G, M)$ are not necessarily of infinite multiplicity, then we have the following theorem.

**THEOREM 3.** Let $a, b \in Z_{\alpha}(G, M)$. If $a \preceq b$, then there is a family $\{v_\gamma\}_{\gamma \in \Gamma}$ of partial isometries in $I(a, b)$ with the following properties:

1. $v_\gamma^*v_\gamma = 0$ if $\gamma \neq \lambda$;
2. $\sum_{\gamma \in \Gamma} v_\gamma v_\gamma^* = q$;
3. $I(a, b) = \sum_{\gamma \in \Gamma} v_\gamma M^b$, 

i.e. each $x \in I(a, b)$ can be written as $\sum_{\gamma \in \Gamma} v_\gamma x_\gamma$ for some $x_\gamma \in M^b$, where the sum converges in the $\sigma$-weak operator topology. In this case, we have $a(t) = \sum_{\gamma \in \Gamma} v_\gamma b(t)\alpha_t(v_\gamma^*)$.

**PROOF.** By Zorn's lemma, there exists a maximal family $\{v_\gamma\}_{\gamma \in \Gamma}$ of partial isometries in $I(a, b)$ such that $v_\gamma^*v_\lambda = 0$ ($\gamma \neq \lambda$). Then we can prove that $\sum_{\gamma \in \Gamma} v_\gamma v_\gamma^* = q$. Assume that $q_0 = q - \sum_{\gamma \in \Gamma} v_\gamma v_\gamma^* \neq 0$. Then, by the definition of $q$, there exists a partial isometry $v$ in $I(a, b)$ such that $vv^*q_0 \neq 0$. By the Comparability Theorem, there are a central projection $z$ in $M^a$ and partial isometries $u_1, u_2$ in $M^a$ such that $u_1^*u_1 = zq_0$, $u_1u_1^* \leq zzv^*v$, $u_2^*u_2 = (q - z)vv^*$, and $u_2u_2^* \leq (q - z)q_0$. Then we have either $u_1 \neq 0$ or $u_2 \neq 0$. If $u_1 \neq 0$, then we set $v_1 = u_1^*v$. Hence

$$v_1v_1^* = u_1^*zv^*zu_1 = u_1^*u_1u_1^* = zq_0 \leq q.$$ 

Similarly, if $u_2 \neq 0$, then we put $v_2 = u_2(q - z)v$ in both cases, we have a contradiction. Thus $q = \sum_{\gamma \in \Gamma} v_\gamma v_\gamma^*$. Further, for each $x \in I(a, b)$, $x = qx = \sum_{\gamma \in \Gamma} v_\gamma x_\gamma$. Since $v_\gamma^*x \in I(a, b)\ast I(a, b) \subseteq I(b, b) = M^b$, put $x_\gamma = v_\gamma^*x$. Then $x = \sum_{\gamma \in \Gamma} v_\gamma x_\gamma$ and $I(a, b) = \sum_{\gamma \in \Gamma} v_\gamma M^b$. Finally, we have

$$\sum_{\gamma \in \Gamma} v_\gamma b(t)\alpha_t(v_\gamma^*) = \sum_{\gamma \in \Gamma} a(t)\alpha_t(v_\gamma)\alpha_t(v_\gamma^*) = \sum_{\gamma \in \Gamma} a(t)\alpha_t(v_\gamma v_\gamma^*) = a(t)\alpha_t(a(1)) = a(t)a(t)^*a(t) = a(t).$$

This completes the proof.

Next we study the special case of Theorem 3.

**THEOREM 4.** Let $M$ be a finite von Neumann algebra and $a, b \in Z_{\alpha}(G, M)$. Suppose that the center $Z(M^b)$ of $M^b$ is contained in the center $Z(M)$ of $M$. If $a \sim b$, then $a \simeq b$.

**PROOF.** Consider a maximal family $\{u_\gamma\}_{\gamma \in \Gamma}$ of partial isometries of $I(a, b)$ such that $u_\gamma u_\gamma^*$ are mutually orthogonal and $u_\gamma^*u_\gamma$ are mutually orthogonal. Put $v = \sum_{\gamma \in \Gamma} u_\gamma$. Then $v$ is a partial isometry of $I(a, b)$. Since $q = a(1)$, suppose that $q - vv^* \neq 0$. From the definition of $q$, as in the proof of Theorem 3, there exists a partial isometry $v_1$ in $I(a, b)$ such that $v_1v_1^* \leq q - vv^*$. Since $M$ is a finite von Neumann algebra, it is clear that $p = q = b(1)$. Let $T$ (resp. $T_0$) be the center valued trace of $M$ (resp. $M^b$). Since $Z(M^b) \subset Z(M)$, the restriction of $T$ to $M^b$ equals $T_0$. Hence we have

$$T_0(q - vv^*) = T(q - vv^*) = T(q - q) = T(v_1v_1^*) = T(v_1^*v_1).$$

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By [5, p. 314, Corollary 2.8], $v_1^*v_1 \preceq q - v^*v$ in $M^b$. Thus there is a partial isometry $u$ in $M^b$ such that $u^*u = v_1^*v_1$ and $uu^* \leq q - v^*v$. Put $v_2 = v_1u^*$. Then $v_2^*v_2 = u^*v_1^*v_1 u^* = uu^*uu^* \leq q - v^*v$ and $v_2^*v_2 = v_1 u^*v_1 u^* = v_1^*v_1 \leq q - vv^*$. Since $v_2$ is a nonzero partial isometry in $I(a,b)$, we have a contradiction. Thus $vv^* = v^*v = q$. Then $I(a,b) = vM^b$ and so $a \simeq b$. This completes the proof.

**COROLLARY 5.** Let $M$ be a finite von Neumann algebra and $a \in Z_\alpha(G,M)$. If $M^\alpha$ is a factor, then $a \dagger 1$ or $a \preceq 1$. Further, if $a$ is a unitary cocycle of $M$, then $a \dagger 1$ or $a \preceq 1$.

Finally we consider the form of invariant subspaces. We refer the reader to [2] for the definitions and notations about invariant subspaces. Let $M$ be a von Neumann algebra acting on the noncommutative Lebesgue space $L^2(M)$ in the sense of Haagerup (see [6]). Let $\{\alpha_t\}_{t \in \mathbb{R}}$ be a $\sigma$-weakly continuous, one-parameter group of $^*$-automorphisms of $M$. Then there is a uniquely unitary group $\{U_t\}_{t \in \mathbb{R}}$ on $L^2(M)$ such that $R_{\alpha_t}(x) = U_t,R_x U_t^*$ for all $x \in M$ and $t \in \mathbb{R}$. In this note, we consider the version of right-invariant subspaces. By [2, Theorem 3.1], we have the following theorem.

**THEOREM 5.** Let $\mathcal{M}$ be a right-pure, right-invariant subspace of $L^2(M)$ that is left-normalized (resp. right-normalized). Then there are a projection $p$ in $M$, a strongly continuous unitary group $\{V_t\}_{t \in \mathbb{R}}$ of $\mathbb{R}$ on $L_p L^2(M)$, and $a \in Z_\alpha(G,M)$ such that

1. $R_{\alpha_t}(x)L_p = V_t R_x V_t^*$ for all $x \in M$, $t \in \mathbb{R}$;
2. $V_t = L_{a(t)} U_t$ for all $t \in \mathbb{R}$;
3. $\mathcal{M} = F[0,\infty)L_p L^2(M)$ (resp. $\mathcal{M} = F(0,\infty)L_p L^2(M)$), where $F$ is the spectral measure for $V$ on $L_p L^2(M)$.

Let $\mathcal{M}$ be a left-normalized, right-pure, right-invariant subspace of $L^2(M)$ with $a \in Z_\alpha(\mathbb{R},M)$. If $a \preceq 1$, by Theorem 3, then there exists a family $\{v_\gamma\}_{\gamma \in \Gamma}$ of partial isometries in $I(a,1)$ such that $a(t) = \sum_{\gamma \in \Gamma} v_\gamma \alpha_t(v_\gamma^*)$. Then we have

$$\sum_{\gamma \in \Gamma} L_{v_\gamma} U_t L_{v_\gamma}^* = \sum_{\gamma \in \Gamma} L_{v_\gamma} L_{\alpha_t(v_\gamma^*)} U_t = L_{a(t)} U_t = V_t.$$  

By the uniqueness of spectral decomposition, we have

$$F(\lambda, \infty) = \sum_{\gamma \in \Gamma} L_{v_\gamma} P(\lambda, \infty) L_{v_\gamma}^*,$$

in particular, $F[0,\infty) = \sum_{\gamma \in \Gamma} L_{v_\gamma} P[0,\infty)L_{v_\gamma}^*$, where $P$ is the spectral measure of $\{U_t\}_{t \in \mathbb{R}}$. Hence

$$\mathcal{M} = F[0,\infty)L^2(M) = \sum_{\gamma \in \Gamma} L_{v_\gamma} P[0,\infty)L_{v_\gamma}^* L^2(M)$$

$$= \sum_{\gamma \in \Gamma} L_{v_\gamma} P[0,\infty)L_{v_\gamma}^* L_{v_\gamma} L^2(M) = \sum_{\gamma \in \Gamma} L_{v_\gamma} L_{v_\gamma}^* L_{v_\gamma} P[0,\infty)L^2(M)$$

$$= \sum_{\gamma \in \Gamma} L_{v_\gamma} P[0,\infty)L^2(M) = \sum_{\gamma \in \Gamma} \oplus L_{v_\gamma} H^2,$$

because $v_\gamma^*v_\gamma \in M^\alpha$ and $P[0,\infty) \in L(M^\alpha)'$. Thus we have the following proposition.
PROPOSITION 6. Let \( M \) be a left-normalized (resp. right-normalized), right-pure, right-invariant subspace of \( L^2(M) \) with \( a \in Z_\alpha(R, M) \). If \( a \geq 1 \), then there exists a family \( \{v_\gamma\}_{\gamma \in \Gamma} \) of partial isometries of \( I(a, 1) \) such that \( M = \sum_{\gamma \in \Gamma} \oplus L_{v_\gamma} H^2 \) (resp. \( \sum_{\gamma \in \Gamma} \oplus L_{v_\gamma} H^3 \)).

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