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POSITIVE p-SUMMING OPERATORS ON L_p-SPACES

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ABSTRACT. It is shown that for any Banach space B every positive p-summing operator from $L^{p'}(\mu)$ in B, $1/p + 1/p' = 1$, is also cone absolutely summing. We also prove here that a necessary and sufficient condition that B has the Radon-Nikodým property is that every positive p-summing operator $T\colon L^{p'}(\mu) \to B$ is representable by a function f in $L^p(\mu, B)$.

1. Introduction. In this paper we shall be concerned with a weaker concept than a p-absolutely summing operator [**5**] and stronger than a p-concave one [**4**]. This concept makes sense when we are dealing with operators T in $L(X, B)$, with X a Banach lattice. An operator which maps positive sequences $\{x_n\}$ with $\sup_{\|\xi\|_{X^*} \leq 1} \sum |\langle \xi, x_n\rangle|^p < \infty$ in sequences $\{Tx_x\}$ such that $\sum \|Tx_n\|^p < \infty$ will be called a positive p-summing operator.

In case $p = 1$, such operators are called order summing [**2**] or cone absolutely summing operators [**7**] and for $1 < p < \infty$ they have already been considered by the author in [**1**]. Here we shall investigate the space of positive p-summing operators for spaces $C(\Omega)$ and $L^r(\mu)$ $(1 \leq r < \infty)$. We shall find that for any Banach space B the positive p-summing operators from $L^{p'}(\mu)$ in B denoted by $\Lambda_p(L^{p'}(\mu), B)$, with $1/p + 1/p' = 1$, are also cone absolutely summing ones.

We shall obtain a necessary and sufficient condition such that B has the Radon-Nikodým property in terms of these operators. This condition can be written as follows:

$$\Lambda_p(L^{p'}(\mu), B) = L^p(\mu, B) \quad \text{for some } p,\ 1 < p \leq \infty.$$

2. Definitions and preliminary results. Throughout this paper X will denote a Banach lattice, B a Banach space, and $L(X, B)$ the space of bounded operators from X into B. We shall write p' for the number such that $1/p + 1/p' = 1$.

DEFINITION 1. Let $1 \leq p < \infty$. An operator $T\colon X \to B$ is said to be positive p-summing if there exists a constant $C > 0$ such that for every $x_1, x_2, \ldots, x_n$, *positive* elements in X, we have

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \leq C \sup_{\|\xi\|_{X^*} \leq 1} \left(\sum_{i=1}^{n} |\langle \xi, x_i\rangle|^p\right)^{1/p}. \tag{1}$$

We shall denote by $\Lambda_p(X, B)$ the space of positive p-summing operators. This space becomes a Banach space with the norm $\| \ \|_{\Lambda_p}$ given by the infimum of the constants verifying (1).

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For $p=\infty$ we consider $\Lambda_\infty(X,B)=L(X,B)$ and $\|T\|_{\Lambda_\infty}=\|T\|$.

We shall also denote by $\Pi_p(X,B)$ and $\mathcal{C}_p(X,B)$ the spaces of p-absolutely summing and p-concave operators respectively (see [**5**, **4**]).

A simple use of the duality $(l^p)^*=l^{p'}$ leads us to the following useful equality:

$$\sup_{\|\xi\|_{X^*}\leq 1}\left(\sum_{i=1}^n |\langle \xi, x_i\rangle|^p\right)^{1/p} = \sup_{\alpha\in U^+_{p'}}\left\|\sum_{i=1}^n \alpha_i x_i\right\|_X, \tag{2}$$

where

$$U^+_{p'}=\left\{\alpha=(\alpha_i)_{i=1}^n\colon \sum_{i=1}^n|\alpha_i|^{p'}\leq 1,\ \alpha_i\geq 0\right\}.$$

The first fact we shall notice is the relationship between these three types of operators.

PROPOSITION 1.

$$\Pi_p(X,B)\subsetneqq \Lambda_p(X,B)\subsetneqq \mathcal{C}_p(X,B)\qquad (1\leq p\leq\infty).$$

PROOF. The first inclusion is completely obvious. To see the second one, let us take T in $\Lambda_p(X,B)$ and $x_1,x_2,\dots,x_n$ in X,

$$\begin{aligned}\left(\sum_{i=1}^n\|Tx_i\|_B^p\right)^{1/p} &\leq \left(\sum_{i=1}^n\|Tx_i^+\|_B^p\right)^{1/p}+\left(\sum_{i=1}^n\|Tx_i^-\|_B^p\right)^{1/p}\\ &\leq \|T\|_{\Lambda_p}\left(\sup_{\alpha\in U^+_{p'}}\left\|\sum_{i=1}^n\alpha_i x_i^+\right\|_X+\sup_{\alpha\in U^+_{p'}}\left\|\sum_{i=1}^n\alpha_i x_i^-\right\|_X\right)\\ &\leq 2\|T\|_{\Lambda_p}\sup_{\alpha\in U^+_{p'}}\left\|\sum_{i=1}^n\alpha_i|x_i|\right\|_X.\end{aligned}$$

By using the homogeneous functional calculus in a lattice given by Krivine we have that

$$\sum_{i=1}^n\alpha_i|x_i|\leq\left(\sum_{i=1}^n|x_i|^p\right)^{1/p}\qquad\text{for all } \alpha\in U^+_{p'},$$

and then

$$\left(\sum_{i=1}^n\|Tx_i\|_B^p\right)^{1/p}\leq 2\,\|T\|_{\Lambda_p}\left\|\left(\sum_{i=1}^n|x_i|^p\right)^{1/p}\right\|_X.$$

So $T\in\mathcal{C}_p(X,B)$. $\square$

REMARKS. Let us give two examples to realize that these inclusions may be strict.

In Proposition 3 below we shall prove that $\Lambda_p(L^1(\mu),B)=L(L^1(\mu),B)$ for all p, $1\leq p\leq\infty$. On the other hand if we consider B a Banach space without the Radon-Nikodým property then there will exist an operator $T\colon L^1(\mu)\to B$ which is not representable by a function (see [**2**]). Therefore this operator T cannot belong to $\Pi_p(L^1(\mu),B)$ since every p-absolutely summing operator is weakly compact and these last ones are always representables (see [**2**, p. 75]).

An example of a p-concave operator and a nonpositive p-summing one may simply be the identity $I: l^p \to l^p$ for $1 < p \le 2$. This fact can be shown by taking $\{e_n\}$ as the usual basis in l^p and by noticing that

$$\left(\sum_{i=1}^{n} \|e_i\|^p\right)^{1/p} = n^{1/p}$$

and

$$\sup_{\|\xi\|_{p'} \le 1} \left(\sum_{i=1}^{n} |\langle \xi, e_i \rangle|^p\right)^{1/p} = \sup_{\|\xi\|_{p'} \le 1} \|\xi\|_p \le \sup_{\|\xi\|_{p'}=1} \|\xi\|_{p'} \le 1.$$

We state here two facts which we shall use later. Part (3) is immediate and part (4) can be proved by standard arguments.

PROPOSITION 2.

(3) *If* $X_1 \subseteqq X_2$, $\overline{X}_1 = X_2$, *and* $1 \le p \le \infty$, *then* $\Lambda_p(X_2, B) \subseteqq \Lambda_p(X_1, B)$,

(4) $$\Lambda_p(X, B) \subseteqq \Lambda_q(X, B) \quad \textit{if } 1 \le p \le q \le \infty.$$

3. The main results. In this section we shall denote by Ω a compact space and by $(\Omega, \mathcal{A}, \mu)$ a finite measure space. We shall write $L^p(\mu, B)$ for the space of measurable functions on Ω with

$$\|f\|_p = \left(\int_\Omega \|f(t)\|^p \, d\mu\right)^{1/p} < \infty.$$

The p-absolutely summing operators for L^r-spaces have been considered by several authors, for instance in [**4**, **6**]. Here we shall study the positive p-summing ones, obtaining some analogous results.

PROPOSITION 3. *Let* $1 \le p \le \infty$.

(5) $$\Pi_p(C(\Omega), B) = \Lambda_p(C(\Omega), B) = \mathcal{C}_p(C(\Omega), B),$$

(6) $$\Lambda_p(L^1(\mu), B) = L(L^1(\mu), B).$$

PROOF We can obtain (5) as an easy consequence of the following fact.
For $\psi_1, \psi_2, \ldots, \psi_n$ belonging to $C(\Omega)$, by using (2) we have

$$\begin{aligned}
\left\| \left(\sum_{i=1}^{n} |\psi_i|^p\right)^{1/p} \right\|_{C(\Omega)} &= \sup_{t\in\Omega} \left(\sum_{i=1}^{n} |\psi_i(t)|^p\right)^{1/p} \\
&= \sup_{t\in\Omega} \sup_{\alpha\in U_{p'}^+} \left|\sum_{i=1}^{n} \psi_i(t)\alpha_i\right| \\
&= \sup_{\alpha\in U_p^+} \left\|\sum_{i=1}^{n} \alpha_i\psi_i\right\|_{C(\Omega)}.
\end{aligned}$$

From (3) it suffices to show that $\Lambda_1(L^1(\mu), B) = L(L^1(\mu), B)$ to see (6). Now given $\psi_1, \psi_2, \ldots, \psi_n \geq 0$ in $L^1(\mu)$ and T an operator in $L(L^1(\mu), B)$ we have

$$\sum_{i=1}^{n} \|T(\psi_1)\|_B \leq \|T\| \sum_{i=1}^{n} \|\psi_i\|_1 = \|T\| \cdot \left\| \sum_{i=1}^{n} \psi_i \right\|_1$$
$$= \|T\| \sup_{\substack{\varphi \in L^\infty(\mu) \\ \|\varphi\|_\infty \leq 1}} \sum_{i=1}^{n} |\langle \varphi, \psi_i \rangle|.$$

Therefore T belongs to $\Lambda_1(L^1(\mu), B)$. $\square$

THEOREM 1.

$$\Lambda_p(L^{p'}(\mu), B) = \Lambda_1(L^{p'}(\mu), B) \quad \textit{for } 1 \leq p \leq \infty. \tag{7}$$

PROOF. Cases $p = 1$ and $p = \infty$ are already proved. Let us suppose $1 < p < \infty$ and let us take T in $\Lambda_p(L^{p'}(\mu), B)$. We are going to see that T belongs to $\Lambda_1(L^{p'}(\mu), B)$.

Let us consider the finitely additive measure $G\colon \mathcal{A} \to B$ defined by $G(E) = T(\chi_E)$ for all measurable sets E. It is easy to verify that G is countably additive. Now given E in $\mathcal{A}$ and denoting by π_E the finite partitions of E, by Hölder's inequality and from (2) we have the following:

$$\begin{aligned}
|G|(E) &= \sup_{\pi_E} \sum_{i=1}^{n} \|G(A_i)\| \\
&= \sup_{\pi_E} \sum_{i=1}^{n} \|T(\mu(A_i)^{-1/p'} \cdot \chi_{A_i})\| \mu(A_i)^{1/p'} \\
&= \sup_{\pi_E} \left(\sum_{i=1}^{n} \left\| T(\mu(A_i)^{-1/p'} \chi_{A_i}) \right\|^p \right)^{1/p} \cdot \mu(E)^{1/p'} \\
&\leq \mu(E)^{1/p'} \cdot \|T\|_{\Lambda_p} \sup_{\alpha \in U_{p'}^+} \left\| \sum_{i=1}^{n} \alpha_i \mu(A_i)^{-1/p'} \cdot \chi_{A_i} \right\|_{p'} \\
&= \|T\|_{\Lambda_p} \cdot \mu(E)^{1/p'}.
\end{aligned}$$

From this it follows that $|G|$ is a finite positive measure which is absolutely continuous with respect to μ. Therefore the Radon-Nikodým theorem implies that there exists a function $g \geq 0$ in $L^1(\mu)$ with $|G|(E) = \int_E g(t)\, d\mu$ for all E in $\mathcal{A}$. Let us prove that g belongs to $L^p(\mu)$. Indeed, since

$$\|g\|_p = \sup \left\{ \left| \int_\Omega g(t) s(t)\, d\mu \right| : s = \sum_{i=1}^{n} \alpha_i \chi_{E_i},\ \|s\|_{p'} \leq 1 \right\},$$

then it is clear that

$$\|g\|_p \leq \sup \left\{ \sum_{i=1}^{n} |G|(E_i) \cdot \alpha_i : \sum_{i=1}^{n} \alpha_i^{p'} \mu(E_i) \leq 1,\ \alpha_i \geq 0 \right\}.$$

By checking this sum we have

$$\sum_{i=1}^{n} |G|(E_i)\alpha_i = \sum_{i=1}^{n} \left(\sup_{\pi_{E_i}} \sum_{i=1}^{j_i} \|G(E_{i,j})\| \right) \alpha_i$$
$$\leq \sup_{\pi_\Omega} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{j_i} \|G(E_{i,j})\| \alpha_{i,j}, \sum_{i,j} \alpha_{i,j}^{p'} \mu(E_{i,j}) \leq 1 \right\}.$$

So we obtain that

$$\|g\|_p \leq \sup \left\{ \sum_{k=1}^{m} \|G(A_k)\| \cdot \beta_k \colon \sum_{k=1}^{m} \beta_k^{p'} \mu(A_k) \leq 1,\ m \in \mathbf{N}, \beta_k \geq 0 \right\}$$
$$= \sup \left\{ \sum_{i=1}^{m} \|T(\mu(A_k)^{-1/p'} \chi_{A_k})\| \cdot \gamma_k, \sum_{k=1}^{m} \gamma_k^{p'} \leq 1,\ m \in \mathbf{N}, \gamma_k \geq 0 \right\}$$
$$\leq \sup_{m \in \mathbf{N}} \left(\sum_{i=1}^{m} \|T(\mu(A_k)^{-1/p'} \cdot \chi_{A_k}\|^p \right)^{1/p} \leq \|T\|_{\Lambda_p}.$$

From this $\|g\|_p = \|T\|_{\Lambda_p}$.

Now since $\|T(\chi_E)\| \leq \int_\Omega \chi_E \cdot g(t)\, d\mu$ we can obtain

$$\|T(\psi)\| \leq \int_\Omega |\psi(t)| g(t)\, d\mu \quad \text{for all } \psi \in L^{p'}(\mu). \tag{8}$$

From (8) it is easy to verify that T belongs to $\Lambda_1(L^{p'}(\mu), B)$. Indeed, given $\psi_1, \psi_2, \ldots, \psi_n \geq 0$ in $L^{p'}(\mu)$ we have

$$\sum_{i=1}^{n} \|T(\psi_i)\| \leq \sum_{i=1}^{n} \int_\Omega \psi_i(t) g(t)\, d\mu$$
$$= \int_\Omega g(t) \left(\sum_{i=1}^{n} \psi_i(t) \right) d\mu \leq \|g\|_p \cdot \left\| \sum_{i=1}^{n} \psi_i \right\|_{L^{p'}(\mu)}.$$

Therefore $\|T\|_{\Lambda_p} = \|T\|_{\Lambda_1}$, and this finished the proof. □

Let us remark that Rosenthal's result [**6**] together with the fact, proved by Lindenstrauss and Pełczyński in [**3**], that $\Pi_2(C(\Omega), L^p(\mu)) = L(C(\Omega), L^p(\mu))$ for $1 \leq p \leq 2$ allow us to state that for any Banach space B

$$\Pi_2(L^p(\mu), B) = \Pi_1(L^p(\mu), B) \quad \text{for } 1 \leq p \leq 2. \tag{9}$$

This result has an analogue in our context.

COROLLARY 1. *If $1 \leq p \leq 2$, then*

$$\Lambda_2(L^p(\mu), B) = \Lambda_1(L^p(\mu), B).$$

Let us recall that B has the Radon-Nikodým property if and only if every operator T in $L(L^1(\mu), B)$ is representable by a function f in $L^\infty(\mu, B)$; in our terminology,

$$\Lambda_\infty(L^1(\mu), B) = L^\infty(\mu, B). \tag{10}$$

This result can be extended for every value of p.

First of all, every function f in $L^p(\mu, B)$ determines an operator $L\colon L^{p'}(\mu) \to B$ given by

$$T(\psi) = \int_\Omega f(t)\psi(t)\,d\mu.$$

It is simple computation to verify that T belongs to $\Lambda_1(L^{p'}(\mu), B)$ and therefore to $\Lambda_p(L^{p'}(\mu), B)$. This means that $L^p(\mu, B) \subseteqq \Lambda_p(L^{p'}(\mu), B)$. In addition we have the following

THEOREM 2. *Let $1 < p \leq \infty$. The following are equivalent:*
(a) *B has the Radon-Nikodým property.*
(b) $\Lambda_p(L^{p'}(\mu), B) = L^p(\mu, B)$.

PROOF. Let us suppose B has the Radon-Nikodým property and let us take T in $\Lambda_p(L^{p'}(\mu), B)$.

By considering $G(E) = T(\chi_E)$ we proved in Theorem 1 that G is a measure absolutely continuous with respect to μ and with bounded variation. The Radon-Nikodým property of B implies the existence of a function f in $L^1(\mu, B)$ such that $G(E) = \int_E f(t)\,d\mu$.

Therefore $|G|(E) = \int_E \|f(t)\|\,d\mu$ and as in the demonstration of Theorem 1 it can be shown that f belongs to $L^p(\mu, B)$ and, besides, T is representable by f.

To see the converse let us suppose $\Lambda_p(L^{p'}(\mu), B) = L^p(\mu, B)$ and let us take an operator T in $L(L^1(\mu), B)$. From (6) and (3) we have T in $\Lambda_p(L^{p'}(\mu), B)$ and therefore T is representable by a function in $L^p(\mu, B)$, this is $T(\psi) = \int_\Omega \psi(t)f(t)\,d\mu$ for every simple function ψ.

Finally a standard argument shows that f actually belongs to $L^\infty(\mu, B)$ and $T(\psi) = \int_\Omega f(t)\psi(t)\,d\mu$ for all ψ in $L^1(\mu)$. □

REFERENCES

1. O. Blasco, *Boundary values of vector valued harmonic functions considered as operators*, Studia Math. (to appear).
2. J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R. I., 1977.
3. J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, Studia Math. **29** (1968), 241–252.
4. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, vol. II, Springer-Verlag, Berlin and New York, 1979.
5. A. Pietsch, *Absolut p-summierende Abbildungen in normirten Raumen*, Studia Math. **28** (1967), 333–353.
6. H. P. Rosenthal, *On subspaces of L^p*, Ann of Math. (2) **97** (1973), 344–373.
7. H. H Schaeffer, *Banach lattices and positive operators*, Springer-Verlang, Berlin and New York, 1974.

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