ON n-DIMENSIONAL LORENTZ MANIFOLDS ADMITTING
AN ISOMETRY GROUP OF DIMENSION n(n - 1)/2 + 1

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ABSTRACT. We classify connected n-dimensional Lorentz manifolds admitting an isometry group of dimension n(n - 1)/2 + 1 with compact isotropy subgroup (n ≥ 5).

1. A connected n-dimensional Riemannian manifold admitting a connected closed isometry group of dimension n(n - 1)/2 + 1 (n ≥ 4) was completely determined by Yano [8], Ishihara [1], and Obata [5] (cf. Kobayashi [2]). We consider the classification problem of Lorentz manifolds. Each of the following examples is a connected n-dimensional Lorentz manifold M admitting a connected isometry group G of dimension n(n - 1)/2 + 1.

EXAMPLE. (i) M = R x N with metric -dt^2 + ds^2_N, and G = R x I^0(N).
(ii) M = S^1 x N with metric -dθ^2 + ds^2_N, and G = S^1 x I^0(N).
(iii) M = R x P with metric -dt^2 + ds^2_P, and G = R x I^0(P).
(iv) M = S^1 x P with metric -dθ^2 + ds^2_P, and G = S^1 x I^0(P).
(v) M = U^+ = {(u_1,..., u_n); u_n > 0} with metric

|ds^2|^2 = (du_1^2 + ... + du_{n-1}^2 - du_n^2)/(cu_n)^2 (c ≠ 0),

and G = I^0(U^+^+) (see Nomizu [4]).
(vi) M = U^- = {(u_1,..., u_n); u_n > 0} with metric

|ds^2|^2 = (-du_1^2 + du_2^2 + ... + du_n^2)/(cu_n)^2 (c ≠ 0),

and G = I^0(U^-^-) (see Matsuda [3]).

Here N is a simply connected (n - 1)-dimensional Riemannian manifold with metric ds^2_N of constant curvature and P is an (n - 1)-dimensional real projective space with standard metric ds^2_P. A real line and a circle of a certain radius are denoted by R and S^1 respectively. I^0(·) denotes the identity component of the full-isometry group of (·).

The purpose of this note is to prove the following theorem.

THEOREM. Let M be a connected n-dimensional Lorentz manifold admitting a connected isometry group G of dimension n(n - 1)/2 + 1 (n ≥ 5) whose isotropy subgroup at every point is compact. Then M must be one of the spaces (i)-(v).

REMARK 1. The isotropy subgroup of G in the above example is compact except (vi).

REMARK 2. The spaces of (v) and (vi) are not geodesically complete.
2. Let \((M, \langle \cdot, \cdot \rangle)\) be a connected \(n\)-dimensional Lorentz manifold with signature \((-,+,...,+).\) Let \(G\) be a connected isometry group of \((M, \langle \cdot, \cdot \rangle)\) and \(H_p\) the isotropy subgroup of \(G\) at a point \(p \in M.\) Then the linear isotropy group \(\tilde{H}_p = \{dh_p; h \in H_p\}\) acting on \(T_p M\) is a closed subgroup of \(O(1, n - 1).\)

**LEMMA 1.** Every compact subgroup of \(O(1, n - 1)\) is conjugate to a subgroup of \(O(1) \times O(n - 1)\) (cf. Wolf [7]). Especially if \(K\) is a compact subgroup of \(O(1, n - 1)\) whose dimension is \((n - 1)(n - 2)/2\), then \(K\) leaves invariant one and only one-dimensional subspace in an \(n\)-dimensional vector space (cf. Obata [5]).

We can see that for \(n(n+1)/2 > r > n(n-1)/2 + 1\) the full-isometry group of \(M\) contains no subgroup of \(r\) whose isotropy subgroup is compact. In fact, suppose that for \(n(n+1)/2 > r > n(n-1)/2 + 1\) there exists an \(r\)-dimensional subgroup \(G\) whose isotropy group \(H_p\) is compact for some \(p \in M.\) Then \(\dim H_p \leq (n-1)(n-2)/2\) from Lemma 1. On the other hand, we have

\[
\dim H_p \geq \dim G - \dim M > n(n-1)/2 + 1 - n = (n-1)(n-2)/2
\]

which is a contradiction to the fact that \(\dim H_p \leq (n-1)(n-2)/2.\)

We can also derive the following proposition from Lemma 1.

**PROPOSITION.** If \(M\) admits a connected isometry group \(G\) of dimension \(n(n-1)/2 + 1\) whose isotropy subgroup \(H_p\) is compact for every \(p \in M,\) then \(G\) is transitive on \(M.\)

**PROOF.** Assume that \(G\) is not transitive on \(M.\) Then the orbit of \(G\) through \(p\) is of dimension less than \(n.\) Hence,

\[
\dim H_p \geq \dim G - (n-1) = (n-1)(n-2)/2 + 1
\]

which is impossible from Lemma 1.

Hereafter, let \(G\) be a connected isometry group of dimension \(n(n-1)/2 + 1\) whose isotropy subgroup \(H_p\) is compact for every \(p \in M.\) From the proposition, \(\dim H_p = (n-1)(n-2)/2\) and we may assume that \(M = G/H_p.\)

Let \(\mathfrak{g}\) and \(\mathfrak{h}\) be the Lie algebras of \(G\) and \(H := H_0\) respectively. But the use of an \(\text{Ad}(H)\)-invariant positive definite inner product on \(\mathfrak{g}\) which exists from the compactness of \(H,\) we have a decomposition \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\) (direct sum) of \(\mathfrak{g}\) where \(\text{Ad}(H)m \not\subseteq \mathfrak{m}.\) Let \(\pi: G \to G/H\) be the natural projection. We identify the tangent space \(T_o M\) at \(o := \pi(H)\) and \(\mathfrak{m}\) by \(d\pi.\) Then the linear isotropy group \(\tilde{H}\) acting on \(T_o M\) corresponds under \(d\pi\) to \(\text{Ad}(H)\) on \(\mathfrak{m}.\) The Lorentz inner product \(T_o M\) induces the Lorentz inner product \(\langle \cdot, \cdot \rangle_{\mathfrak{m}}\) on \(\mathfrak{m}\) so that \(d\pi: \mathfrak{m} \to T_o M\) is a linear isometry. We note that the inner product \(\langle \cdot, \cdot \rangle_{\mathfrak{m}}\) on \(\mathfrak{m}\) is \(\text{Ad}(H)-\text{invariant}.\)

We define the Lorentz inner product \(B\) on \(\mathfrak{g}\) such that

\[
B(\mathfrak{h}, \mathfrak{m}) = 0, \quad B|_{\mathfrak{m}} = \langle \cdot, \cdot \rangle_{\mathfrak{m}},
\]

and \(B|_{\mathfrak{h}}\) is positive definite. We extend \(B\) to the \(G\)-left invariant Lorentz metric on \(G\) which is denoted by the same letter \(B.\) Then \((G, B)\) is a Lorentz manifold and \(\pi: (G, B) \to (M, \langle \cdot, \cdot \rangle)\) is a semi-Riemannian submersion (see O'Neill [6, p. 212]).

Since \(\text{Ad}(H)\) acts on \(\mathfrak{m}\) as a compact linear isometry group and \(\dim \text{Ad}(H) = (n-1)(n-2)/2,\) it follows from Lemma 1 that there exists one and only one \(1\)-dimensional subspace \(\mathfrak{m}_1\) of \(\mathfrak{m}\) which is invariant by \(\text{Ad}(H).\) Furthermore, we have
Lemma 2. \( m_1 \) is timelike.

Putting \( m_2 = \{ X \in m; B(X, m_1) = 0 \} \), we have a direct sum decomposition \( m = m_1 + m_2 \) of \( m \) where \( [h, m_1] = 0 \), \( [h, m_2] = m_2 \), and the adjoint representation of \( h \) in \( m_2 \) is irreducible. Thus we have a decomposition \( g = m_1 + m_2 + h \) of \( g \) by \( \text{ad}(h) \), where \( [h, m_1] = 0 \) and \( [h, m_2] = m_2 \) (see Obata [5]).

Lemma 3 (Obata [5]). We have the following possibilities provided \( n \geq 5 \):

(i) \( [h, m_1] = 0 \), \( [m_1, m_2] = 0 \), \( [m_2, m_2] = 0 \);
(ii) \( [h, m_1] = 0 \), \( [m_1, m_2] = 0 \), \( [m_2, m_2] = h \);
(iii) \( [h, m_1] = 0 \), \( [m_1, m_2] = m_2 \), \( [m_2, m_2] = 0 \), and \( [X, Y] = L(X)Y \) for any \( X \in m_1 \) and \( Y \in m_2 \) where \( L \) is the linear function on \( m_1 \) such that \( L(X) \neq 0 \) for any nonzero \( X \in m_1 \).

Remark 3. For a unit vector \( E \in m_1 \) (i.e. \( B(E, E) = -1 \)) we put \( c := L(E) \). We may assume \( c > 0 \).

We set \( g' = h + m_2 \) and \( B' = \frac{1}{c} B' \). Since \( B' \) is positive definite and \( \text{ad}(h) |_{m_2} = o(n-1) \), there is a basis \( \{ X_i, X_{jk} \} \) (1 \( \leq i \leq n-1 \), 1 \( \leq j < k \leq n-1 \)) of \( g' \) such that

1. \( \{ X_i \} \) (resp. \( \{ X_{jk} \} \)) is a basis of \( m_2 \) (resp. \( h \)),
2. \( B'(X_k, X_j) = \delta_{ij} \),
3. \( [X_{ij}, X_{kj}] = \delta_{ik} X_j - \delta_{jk} X_i \) (1 \( \leq i < j \leq n-1 \), 1 \( \leq k \leq n-1 \)).

Then

Lemma 4 (Obata [5]). In case (ii) in Lemma 3, there exists a nonzero constant \( \alpha \) such that \( [X_i, X_j] = \alpha X_{ij} \) (1 \( \leq i < j \leq n-1 \)).

3. From Lemma 3, \( m_1 \) (resp. \( m_2 \)) induces an integrable \( G \)-invariant 1- (resp. \( n-1 \)-) dimensional distribution \( T_1 \) (resp. \( T_2 \)) on \( M \) such that at each point \( p \) of \( M \), \( T_p M = T_1(p) + T_2(p) \), \( \langle T_1(p), T_2(p) \rangle = 0 \), and \( T_1(p) \) (resp. \( T_2(p) \)) is timelike (resp. spacelike).

Now, we assume that \( M \) is simply connected.

Lemma 5. When \( M \) is simply connected, \( \xi := d\pi(E) \) is well defined on \( M \).

Proof. We will show that for each \( p \in M \), \( \xi(p) = d\pi(E(g)) = d\tau_g d\pi(E(e)) \) is independent of the choice of \( g \in G \) such that \( g o = p \), where \( e \) is the identity of \( G \), \( o = \pi(H) \), and \( \tau_g \) is the map: \( x \rightarrow gx \) on \( M \).

Let \( g_1 = g_2 o = p \) (\( g_1, g_2 \in G \)). \( G \) being connected, there exist curves \( \tilde{g}_i : [0, 1] \rightarrow G \) such that \( \tilde{g}_i(0) = e \) and \( \tilde{g}_i(1) = g_i \) (\( i = 1, 2 \)). Set \( c_i(t) := \tilde{g}_i(t) o \) (\( i = 1, 2 \)). \( M \) being simply connected, \( M \) is time orientable. So there exists a unit timelike vector field \( X \) on \( M \). Then we can see that

\( \langle X(c_i(t)), d\tau_{\tilde{g}_i(t)} \xi(o) \rangle \neq 0 \) for any \( t \in [0, 1] \).

The map: \( t \rightarrow \langle X(c_i(t)), d\tau_{\tilde{g}_i(t)} \xi(o) \rangle \) being continuous, if \( \langle X(o), \xi(o) \rangle < 0 \) (resp. \( > 0 \)), then \( \langle X(p), d\tau_{\tilde{g}_1} \xi(o) \rangle < 0 \) (resp. \( > 0 \)). Thus \( d\tau_{g_1} \xi(o) \) and \( d\tau_{g_2} \xi(o) \) belong to the same connected component of the time cone in \( T_p M \). Furthermore, \( d\tau_{g_1} \xi(o) \) and \( d\tau_{g_2} \xi(o) \) belong to \( T_1(p) \). Therefore \( d\tau_{g_1} \xi(o) = d\tau_{g_2} \xi(o) \).

We define the nonzero 1-form \( \omega \) on \( M \) by \( \omega(X) := \langle X, \xi \rangle \). Then we can see \( \omega \) is closed. Since \( M \) is simply connected there exists a \( C^\infty \) function \( f : M \rightarrow \mathbb{R} \) such that \( df = \omega \). For each \( a \in f(M) \), \( f^{-1}(a) \) is a closed spacelike hypersurface.
of $M$ and each connected component of $f^{-1}(a)$ is a leaf of $T_2$. Since $\nabla_x \xi = 0$ and $\xi = d\pi(E)$, each integral curve of $\xi$ which is a leaf of $T_1$ is a complete geodesic. For a point $p \in M$, let $\gamma_p(t)$ be the integral curve of $\xi$ such that $\gamma_p(0) = p$. Then we have easily that $f(\gamma_p(t)) = -t + a$ for $p \in f^{-1}(a)$. Therefore $f(M) = \mathbb{R}$. Let $N$ be $f^{-1}(0)$ and $N_0$ be a connected component of $N$. We define the map $F: \mathbb{R} \times N_0 \to M$ by $F(t,x) := \gamma_x(t) = \exp(t\xi(x))$. Then we have

**Lemma 6.** $F$ is the onto diffeomorphism.

**Proof.** Assume that $F(t,x) = F(t',x')$. We have

$$t = -f(\gamma_x(t)) = -f(F(t,x)) = -f(F(t',x')) = t'.$$

Since geodesics $\gamma_x$ and $\gamma_{x'}$ are leaves of $T_1$ through $F(t,x) = F(t',x')$ and $t = t'$, we have $x = x'$. Thus $F$ is one-to-one. It is evident that $F$ is differentiable. Set $M_0 := F(\mathbb{R} \times N_0)$. Then $M_0$ is open in $M$. It remains to be shown that $M_0$ is closed in $M$. Suppose that $F(t_k,x_k) = p_k$ is a sequence approaching some point $q$ in $M$. Let $\tilde{f}: \mathbb{R} \to \mathbb{R}$ be the function defined by $\tilde{f}(t) := f(F(t,x))$ for some $x \in N_0$. Then $\tilde{f}$ is independent of the choice $x \in N_0$, for $\tilde{f}(t) = -t$. Since $\tilde{f}^{-1}(f(p_k)) = t_k$ and $\tilde{f}^{-1}(f(p_k)) \to \tilde{f}^{-1}(f(q))$, we have $t_k \to t_0 := \tilde{f}^{-1}(f(q))$. Letting $x := \gamma_q(-t_0) = \exp(-t_0\xi(q))$, we have

$$x_k = \gamma_{p_k}(-t_k) = \exp(-t_k\xi(p_k)) \to \gamma_q(-t_0).$$

Since $N_0$ is closed, $x$ belongs to $N_0$ so that $q = F(t_0,x) \in M_0$. Thus $M = M_0$; furthermore, $N = N_0$.

Since $M$ is homogeneous and $T_2$ is $G$-invariant, $N$ is a homogeneous Riemannian manifold so that $N$ is complete. Furthermore, $N$ is simply connected, because $M$ which is diffeomorphic to $\mathbb{R} \times N$, is simply connected. Therefore, in cases (i) and (iii) in Lemma 3, $N$ is isometric to the Euclidean space $\mathbb{E}^{n-1}$. In case (ii) in Lemma 3, $N$ is isometric to a sphere $S^{n-1}$ or a hyperbolic space $\mathbb{H}^{n-1}$ by Lemma 4.

Let $(U, \phi = (t_1, \ldots, t_{n-1}))$ be a local coordinate around a point $p$ in $N$. Then $(\mathbb{R} \times U, \text{id} \times \phi = (t,t_1, \ldots, t_{n-1}))$ is a local coordinate around $(a,p)$ in $\mathbb{R} \times N$. Let $\tilde{U} := F(\mathbb{R} \times U)$ and define $\tilde{\phi}: \tilde{U} \to \mathbb{R}^n$ by $(\text{id} \times \phi) \circ F^{-1}$. Then $(\tilde{U}, \tilde{\phi} = (x_0,x_1, \ldots, x_{n-1}))$ is a local coordinate around $\tilde{p} = F(a,p)$ in $M$. We can see that $dF(\partial/\partial t) = \xi = \partial/\partial x_0 \in T_1$ and $dF(\partial/\partial t_i) = \partial/\partial x_i \in T_2$ $(i = 1, \ldots, n - 1)$. So we have $\langle \partial/\partial x_0, \partial/\partial x_0 \rangle = -1$ and $\langle \partial/\partial x_0, \partial/\partial x_i \rangle = 0$ for $1 \leq i \leq n - 1$.

**Lemma 7.** In cases (i) and (ii) in Lemma 3, $F: (\mathbb{R} \times N, -dt^2 + ds_\mathbb{R}^n) \to (M, \langle , \rangle)$ is isometry, where $ds_\mathbb{R}^n$ is the metric of $N$.

**Proof.** It is enough to show that for $1 \leq i, j \leq n - 1$, $\langle \partial/\partial x_i, \partial/\partial x_j \rangle$ is independent of $x_0$. Since $\pi: (G,B) \to (M, \langle , \rangle)$ is the semi-Riemannian submersion, it follows from Lemma 3 that $T_1$ is parallel. So, for $1 \leq i, j \leq n - 1$, we have

$$\langle \partial/\partial x_0, \partial/\partial x_i, \partial/\partial x_j \rangle = \langle \nabla_{\partial/\partial x_0} \partial/\partial x_i, \partial/\partial x_0, \partial/\partial x_j \rangle + \langle \partial/\partial x_i, \nabla_{\partial/\partial x_0} \partial/\partial x_0 \rangle = 0$$

because $\nabla_{\partial/\partial x_0} \partial/\partial x_0$ and $\nabla_{\partial/\partial x_i} \partial/\partial x_0$ belong to $T_1$.  

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lemma 8. In case (iii) in Lemma 3, $F: (\mathbb{R} \times N, -dt^2 + \exp(-2ct)ds^2) \to (M, \langle , \rangle)$ is isometry, where $ds^2$ is the flat metric of $N = \mathbb{E}^{n-1}$.

proof. Since $\pi: (G, B) \to (M, \langle , \rangle)$ is the semi-Riemannian submersion, the equalities $\nabla_{\partial/\partial x_i} \partial/\partial x_0 = -c(\partial/\partial x_i)$ ($1 \leq j \leq n - 1$) follows from Lemma 3. We have

$$(\partial/\partial x_0)(\partial/\partial x_i, \partial/\partial x_j) = -2c(\partial/\partial x_i, \partial/\partial x_j)$$

for $1 \leq i, j \leq n - 1$ so that we have

$$(\partial/\partial x_i, \partial/\partial x_j) = \exp(-2cx_0)g_{ij}(x_1, \ldots, x_{n-1}).$$

Thus

$$F^*\langle , \rangle = -dt^2 + \exp(-2ct) \sum_{i,j=1}^{n-1} g_{ij} \circ F(t_1, \ldots, t_{n-1})dt_i dt_j.$$ 

Since $N$ is isometric to $\mathbb{E}^{n-1}$, we may assume that $g_{ij} = \delta_{ij}$.

Proof of theorem. Consider case (i) in Lemma 3. If $M$ is simply connected, $M$ is isometric to $(\mathbb{R} \times \mathbb{E}^{n-1}, -dt^2 + ds^2)$ by Lemma 7 where $ds^2$ is the metric of $\mathbb{E}^{n-1}$. To find a non-simply-connected $M$, we must look for a discrete subgroup $\Gamma$ of the full isometry group of $\mathbb{R} \times \mathbb{E}^{n-1}$ which commutes with $G = \mathbb{R} \times I^0(\mathbb{E}^{n-1})$ elementwise. It is easy to verify that $\Gamma$ is generated by a translation of $\mathbb{R}$. Hence, if $M$ is not simply connected, then $M$ is isometric to $S^1(r) \times \mathbb{E}^{n-1}$.

Consider case (ii) in Lemma 3. If $M$ is simply connected, $M$ is isometric to $(\mathbb{R} \times \mathbb{R}^{n-1}, -dt^2 + ds^2)$ by Lemma 7 where $N$ is $S^{n-1}(r')$ or $\mathbb{H}^{n-1}(r')$. By the same method as above, if $M$ is not simply connected, then $M$ is isometric to $S^1(r) \times S^{n-1}(r')$, $\mathbb{R} \times \mathbb{P}^{n-1}$, $S^1(r) \times \mathbb{P}^{n-1}$ or $S^1(r) \times \mathbb{H}^{n-1}(r')$.

Consider case (iii) in Lemma 3. If $M$ is simply connected, $M$ is isometric to $(\mathbb{R} \times \mathbb{R}^{n-1}, -dt^2 + \exp(-2ct)\sum_{j=1}^{n-1} dt_j^2)$ by Lemma 8. This space is isometric to the space (v) in the Example by the transformation

$$\mathbb{R} \times \mathbb{R}^{n-1} \ni (t_1, \ldots, t_{n-1}) \to (t_1, \ldots, t_{n-1}, e^{ct}/c) \in U_n^+.$$ 

A discrete subgroup of the full isometry group $I(U_n^+)$ which commutes with $G = I^0(U_n^+)$ elementwise consists of the identity element.

The proof of the theorem is complete.

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References


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