PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 2, June 1987

THE COMPLEMENT OF THE STABLE MANIFOLD FOR ONE DIMENSIONAL ENDOMORPHISMS

CARLOS ARTEAGA

ABSTRACT. Let N denote either the circle S^1 or the closed interval $I = [0, 1]$ and let f be a C^1 endomorphism of N. Let $\Sigma(f)$ be the complement of the union of the stable manifolds of the sinks of f. In this paper we give necessary and sufficient conditions for $\Sigma(f)$ to consist of eventually periodic points.

1. Introduction.

Let N denote either the circle S^1 or the closed interval $I = [0, 1]$ and let $f\colon N \to N$ be a C^1 differentiable map (endomorphism). As usual we say that $x \in N$ is a *periodic point* of f if $f^n(x) = x$ for some $n \geq 1$. In this case we say that x is *hyperbolic* if $|(f^n)'(x)| \neq 1$, a *sink* if $|(f^n)'(x)| < 1$, and a *source* if $|(f^n)'(x)| > 1$. Let $P(f)$ and $P_c(f)$ denote respectively the set of periodic points and the set of sinks. The set $P(f)\backslash P_c(f)$ will be denoted by $P_e(f)$.

Let $x \in P(f)$. The *stable set* of x, $W^s(x)$, is defined as the set of points y such that $\lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0$. Each stable set consists of countably many disjoint (possibly degenerate) intervals. The *stable manifold* of f, $\Delta(f)$, is defined by $\Delta(f) = \bigcup W^s(x)$, where the union is taken over all the sinks of f. We let $\Sigma(f) = N\backslash\Delta(f)$. Note that $\Delta(f)$ and $\Sigma(f)$ are invariant under f.

In studying the dynamics of an endomorphism f of N, the sets $P(f)$ and $\Sigma(f)$ play an important role. In general, $\bigcup_{n=0}^{\infty} f^{-n}(P_e(f)) \subset \Sigma(f)$, but equality need not hold. In this paper we give necessary and sufficient conditions such that $\Sigma(f) = \bigcup_{n=0}^{\infty} f^{-n}(P_e(f))$.

THEOREM. *Let f be a C^1 endomorphism of N. $\Sigma(f) = \bigcup_{n=0}^{\infty} f^{-n}(P_e(f))$ if and only if $P(f)$ is closed and nonempty and for every $x \in P_e(f)$, $W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(\{x\})$.*

COROLLARY. *Let $f\colon N \to N$ be a C^1 endomorphism with all the periodic points hyperbolic. $\Sigma(f) = \bigcup_{n=0}^{\infty} f^{-n}(P_e(f))$ if and only if $P(f)$ is finite and nonempty.*

We remark that the theorem is similar in flavor to results in the same setting which give necessary and sufficient conditions for the nonwandering set to equal the periodic points set [**3**, **5**, **7**, **8**] and for the nonwandering set to equal the chain recurrent set [**2**, **4**]. These results will be the fundamental tools for the proof of the theorem.

2. Proof of the Theorem.

We begin by recalling some basic concepts and establishing preliminary results.

Received by the editors December 26, 1985 and, in revised form, April 9, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54H20.
The author was partially supported by CNPq, Brazil.

Let $f\colon N \to N$ be a C^1 endomorphism of N. For any point $x \in N$, we let $\mathcal{O}_f(x) = \bigcup_{n=-\infty}^{\infty} f^n\{x\}$ and $\mathcal{O}_f^+(x) = \bigcup_{n=0}^{\infty} f^n(x)$. x is said to be *eventually periodic* if $\mathcal{O}_f^+(x)$ is finite or equivalently if some iterate of x is periodic.

The *ω-limit* of a point $x \in N$ is defined by $\omega(x) = \{y \in N; y \in \overline{\mathcal{O}_f^+(x)}\}$. x is said to be *recurrent* if $\omega(x) = \mathcal{O}_f^+(x)$. A point x is called *nonwandering* if for any neighborhood U of x there is an integer $n > 0$ such that $f^n(U) \cap U \neq \varnothing$. Let $R(f)$ and $\Omega(f)$ denote the sets of recurrent and nonwandering points respectively.

Let p be a periodic point of (least) period n which is not a critical point of f^n. Let $k = n$ if f^n preserves orientation at p, and $k = 2n$ if f^n reverses orientation at p. We call p an *expanding periodic point* if there is an open neighborhood V_p of p such that for each $x \in V_p$, $|f^k(x) - p| > |x - p|$.

We say that f has a *horseshoe* if for some n there are disjoint closed intervals J and K such that each J and K f^n-covers both J and K. When f has a horseshoe as above, the f^n-invariant set

$$H = \bigcap_{i=0}^{\infty} f^{-in}(J \cup K)$$

has the full one-sided shift on two symbols as a continuous factor [**1**, **3**].

LEMMA 2.1. *Let f be an endomorphism of N. If $\Sigma(f) = \bigcup_n f^{-n}(P_e(f))$, then f has no horseshoes.*

PROOF. By contradiction, suppose that f has a horseshoe. Then, for some n there is a subset H of N, such that $f^n(H) = H$ and there is a topological semiconjugacy h of $f^n\colon H \to H$ onto the full (one-sided) shift σ on two symbols.

We claim that $R(\sigma) = P(\sigma)$. In fact, let $x \in R(\sigma)$. Choose $y \in H$ such that $h(y) = x$. If $y \in \Delta(f^n/H)$, then there is a sink z of f^n/H such that $\omega(y) = \mathcal{O}_{f^n}^+(z)$ and so $\omega(x) = \mathcal{O}_\sigma^+(h(z))$. This implies that $x \in P(\sigma)$, because $h(z)$ is a periodic point of σ. If $y \in \Sigma(f^n/H)$, then, by hypothesis, y is an eventually periodic point of f^n/H and so x is an eventually periodic point of σ. This implies that $x \in P(\sigma)$. Therefore $R(\sigma) = P(\sigma)$. But this is a contradiction because σ is topologically transitive. So f has no horseshoes.

LEMMA 2.2. *Let f be a C^1 endomorphism of N. Let $y \in P_e(f)\backslash\partial N$ such that $W^s(y) = \mathcal{O}_f(y)$ and $y \in \omega(x)$ for some $x \neq y$. If y is not an expanding periodic point of f^2 then there exists an open interval (a,b) containing y and such that either $(a,y) \cap \mathcal{O}_{f^2}^+(x) = \varnothing$ and f^2 is expanding on (y,b), or $(y,b) \cap \mathcal{O}_{f^2}^+(x) = \varnothing$ and f^2 is expanding on (a,y).*

PROOF. Without loss of generality we may assume that y is a fixed point of f. By hypothesis, there exists an interval (a_1, b_1) containing y and such that $g = f^2$ is strictly monotone increasing on (a_1, b_1).

We will assume that the iterates of x have a subsequence $(f^{n_i}(x))$ in (a_1, y) such that $(f^{n_i}(x))$ is monotonically increasing to y (the proof is similar if the subsequence is contained in (y, b_1)).

We claim that there exists an interval (a,y) such that g is expanding on (a,y). Otherwise, by hypothesis, there exists a sequence (z_n) in (a,y) such that z_n is a fixed point of g and (z_n) approaches y. Hence, there exists n_k such that

$z_n < f^{n_k}(x) < z_m$ for some n and m. Hence $z_n < g^l(f^{n_k}(x)) < z_m$ for every l. This implies that $\lim_{n_i \to \infty} f^{n_i}(x) \neq y$. This is a contradiction and proves that for some interval (a, y), g is expanding on (a, y). Since y is not an expanding point of g, there exists an interval $(y, \tilde{b})$ such that g is not expanding on $(y, \tilde{b})$. Thus by the argument above, there exists an interval (y, b) with $b \le \tilde{b}$ and such that $(y, b) \cap \mathcal{O}_g^+(x) = \varnothing$. This ends the proof of the lemma.

Using the same arguments as Lemma 2.2, one can easily prove the following.

LEMMA 2.3. *Let $y \in P_e(f) \cap \partial I$ such that $W^s(y) = \mathcal{O}_f(y)$ and $y \in \omega(x)$ for some $x \neq y$. Then y is an expanding periodic point of f.*

Now we are ready to prove the theorem. First suppose $\Sigma(f) = \bigcup_{n=0}^{\infty} f^{-n}(P_e(f))$. Since either $\Sigma(f)$ or $\Delta(f)$ is nonempty, so is $P(f)$.

We claim that $\Omega(f) = P(f)$. Otherwise, by Lemma 2.1 [**1**, Theorem A; **3**, Proposition 2; **7**, Lemma 3], any $x \in \Omega(f) \backslash P(f)$ has an infinite further orbit. This is a contradiction, because $x \in \Sigma(f)$ and, by hypothesis, $\Sigma(f)$ consists of eventually periodic points. Hence $\Omega(f) = P(f)$ and this implies that $P(f)$ is closed.

Let $x \in P_e(f)$. We will show that $W^s(x) = \mathcal{O}_f(x)$. Let $y \in W^s(x)$. Since $x \notin \Delta(f)$, $y \in \Sigma(f)$, and consequently y is an eventually periodic point of f. This implies that $y \in \mathcal{O}_f(x)$. Therefore $W^s(x) = \mathcal{O}_f(x)$.

Now suppose that $P(f)$ is closed and nonempty and $W^s(x) = \mathcal{O}_f(x)$ for every $x \in P_e(f)$. We shall show that if $y \in \Sigma(f)$, then y is an eventually periodic point of f. The proof adapts a technique from [**4**, Theorem B].

By contradiction, suppose that y has an infinite further orbit. By [**3**, **6**, **7**, **8**], $\Omega(f) = P(f)$ and consequently $\omega(y) \subset P_e(f)$. Let $y_1 \in \omega(y)$ and let $n_1 = 2j_1$, where j_1 is the period of y_1. We choose a neighborhood I_1 of y_1 as follows.

If y_1 is an expanding periodic point of f^2, we choose $I_1 = (y_1 - \varepsilon_1, y_1 + \varepsilon_1)$ for some $\varepsilon_1 > 0$ such that f^{n_1} is expanding on $I_1 \cap N$.

If y_1 is not an expanding periodic point of f^2, we have, by Lemma 2.3, that $y_1 \notin \partial N$. By Lemma 2.2, we may assume without loss of generality that there is an interval $I_1 = (y_1 - \varepsilon_1, y_1 + \varepsilon_1)$ such that $(y_1 - \varepsilon_1, y_1) \cap \mathcal{O}_{f^2}^+(x) = \varnothing$ and f^{n_1} is expanding on $(y_1, y_1 + \varepsilon_1)$.

Let W_1 and V_1 be neighborhoods of y_1 such that $W_1 \subset V_1 \subset I_1$, $f^{n_1}(W_1) \subset V_1$, and the diameter $l(V_1)$ of V_1 is less than $\varepsilon_1/3$. We claim that there is a point y_2 in $\omega(y) \cap (\overline{V}_1 \backslash \overline{W}_1)$. If y_1 is an expanding point, the claim is obvious. If y_1 is not an expanding point, then by the assumption above, there is $\tilde{y}_2 \in \omega(y)$ such that $\tilde{y}_2 \in W_1 \cap (y_1, y_1 + \varepsilon_1)$. Since f^{n_1} is expanding on $(y_1, y_1 + \varepsilon_1)$, $f^{kn_1}(\tilde{y}_2) \in \overline{V}_1 \backslash \overline{W}_1$ for some positive integer k. This proves the claim.

Let O_1 be the union of neighborhoods of diameter $l(W_1)$ about each periodic point $z \notin W_1$ whose period is at most n_1 and let $K_1 = N \backslash (O_1 \cup W_1)$. Then y_2 is in the interior of K_1. Let $n_2 = 2j_2$, where j_2 is the period of y_2 and note that $n_2 > n_1$. Choose I_2, V_2 and W_2 as above with I_2 contained in the interior of K_1. There is a point y_3 in $\omega(y) \cap (\overline{V}_2 \backslash \overline{W}_2)$. Let O_2 be the union of neighborhoods of diameter $l(W_2)$ about each periodic point $z \notin W_2$ whose period k satisfies $n_1 < k \le n_2$. Let $K_2 = N \backslash (O_1 \cup W_1 \cup O_2 \cup W_2)$ and note that y_3 is in the interior of K_2.

Define K_n inductively as above, and let $G_n = K_n \cap \omega(y)$. Then G_n is a decreasing family of nonempty compact sets, so $\bigcap_{n=1}^{\infty} G_n$ is nonempty. Any point in the intersection is in $\omega(Y)$ but is not periodic, a contradiction.

Hence y is an eventually periodic point of f and this ends the proof of the Theorem.

3. Proof of the Corollary. First suppose that $P(f)$ is finite and nonempty. It is clear that $P(f)$ is closed. By hypothesis $P_e(f)$ consists of sources. This implies that $W^s(x) = \mathcal{O}_f(x)$ for every $x \in P_e(f)$. Hence by the Theorem, $\Sigma(f) = \bigcup_{n=0}^{\infty} f^{-n}(P_e(f))$.

Now suppose $\Sigma(f) = \bigcup_{n=0}^{\infty} f^{-n}(P_e(f))$. By the Theorem, $P(f)$ is closed and nonempty. This implies that $P_e(f)$ is closed because it consists of sources. It follows that there exists $n \in N$ and $c > 1$ such that $|(f^n)'(x)| > c$ for all $x \in P_e(f)$. This implies that $f|_{P_e(f)} \colon P_e(f) \to P_e(f)$ is an expansive homeomorphism. This, together with the fact that $P_e(f)$ is totally disconnected, implies by [**6**, p. 92], that $f|_{P_e(f)}$ is conjugate to a subshift σ of finite type. It follows that all the points in the phase space of σ are periodic. This implies that $P(\sigma)$ is finite and so $P_e(f)$ is finite. It follows that $P_c(f)$ is also finite and therefore $P(f)$ is finite.

References

1. L. Block, *Homoclinic points of mappings of the interval*, Proc. Amer. Math. Soc. **72** (1978), 576–580.
2. L. Block and J. Franke, *The chain recurrent set for maps of the interval*, Proc. Amer. Math. Soc. **87** (1983), 723–727.
3. L. Block, E. M. Coven, I. Mulvey, and Z. Nitecki, *Homoclinic and non-wandering points for maps of the circle*, Ergodic Theory Dynamical Systems **3** (1983), 521–532.
4. L. Block and J. Franke, *The chain recurrent set for maps of the circle*, Proc. Amer. Math. Soc. **92** (1984), 597–603.
5. V. V. Fedorenko and A. N. Sarkovskii, *Continuous mappings of the interval with closed sets of periodic points*, Introduction to Differential and Diffferential-Difference Equations, Kiev, 1980, pp. 137–145. (Russian)
6. R. Mañẽ, *Teoria ergodica*, Projeto Euclides, IMPA, Rio de Janeiro, 1983.
7. Z. Nitecki, *Maps of the interval with closed periodic set*, Proc. Amer. Math. Soc. **85** (1982), 451–456.
8. J.-C. Xiong, *Continuous self-maps of the closed interval whose periodic points form a closed set*, J. China Univ. Sci. Tech. **11** (1981), 14–23.

Departamento de Mathemática, Universidade Federal de São Carlos, CEP. 13560, São Carlos-SP, Brasil