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A REDUCTION OF ALGEBRAIC REPRESENTATIONS OF MATROIDS

BERNT LINDSTRÖM

ABSTRACT. We prove the following result conjectured by M. J. Piff in his thesis (1972).

THEOREM. *Let M be a matroid with an algebraic representation over a field $F(t)$, where t is transcendental over F. Then M has an algebraic representation over F.*

The proof depends on Noether's normalization theorem and the place extension theorem. We obtain the following corollary.

COROLLARY. *If a matroid is algebraic over a field F, then any minor of M is algebraic over F.*

We will assume that the reader is familiar with the foundations of matroid theory (cf. [**3**]). A matroid is called *algebraic over a field* when there is an algebraic representation of M over this field [**3**, p. 184]. M. J. Piff proved in his thesis [**2**, Theorem 3.3], with the aid of the lemma below, that a matroid M, which is algebraic over a field F with prime field P, is algebraic over a transcendental extension $P(t_1, \ldots, t_m)$ of P. He conjectured that M is algebraic over P. This is true by our theorem.

THEOREM. *Let the matroid M be algebraic over a transcendental extension $F(t)$ of F. Then M is algebraic over F.*

We will apply the following lemma [**2**, Lemma 3.2], which is not hard to prove.

LEMMA. *If M is algebraic over $F(\alpha)$, where α is algebraic over F, then M is algebraic over F.*

PROOF OF THE THEOREM. Let M be an algebraic matroid with elements E, a finite subset of a field K. M is assumed to be algebraic over $F(t)$, a subfield of K with t transcendent over F. The transcendence degree of $F(E)$ over $F(t)$ equals r, the rank of M. Let $E = \{e_1, \ldots, e_n\}$.

By Noether's normalization theorem [**1**, Chapter II, Theorem 1] there are elements $y_1, \ldots, y_r$ of K such that each e_i $(1 \leqslant i \leqslant n)$ is integral over $F(t)[y_1, \ldots, y_r]$. Let $E' = E \cup \{y_1, \ldots, y_r\}$ and let M' denote the algebraic matroid of M' over $F(t)$. The matroid M' is an extension of E'. Let $\mathscr{C}$ be the set of all circuits of M'.

We associate variables X_i $(1 \leqslant i \leqslant n + r)$ to the elements of E' such that $e_i \leftrightarrow X_i$ for $1 \leqslant i \leqslant n$ and $y_j \leftrightarrow X_{n+j}$ for $1 \leqslant j \leqslant r$. When $A \subseteq E'$ let X_A denote the

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variables X_k corresponding to the elements of A in some arbitrarily fixed order. For each circuit $C \in \mathscr{C}$ there is an irreducible polynomial $P_C(X_C)$ over $F[t]$ satisfied by C, i.e. $P_C(C) = 0$. All the variables X_i of X_C occur explicitely in $P_C(X_C)$ since C is a circuit.

Let $\bar{F}$ and $\bar{K}$ be the algebraic closures of F and K respectively. We will obtain a ring homomorphism

$$\varphi\colon F[t, y_1, \ldots, y_r] \to \bar{F}[y_1, \ldots, y_r].$$

Define $\varphi(y_i) = y_i$ for $1 \leqslant i \leqslant r$. Then choose $\varphi(t) \in \bar{F}$ distinct from all zeros of coefficients of the polynomials P_C, $C \in \mathscr{C}$ (by assumption the coefficients belong to $F[t]$). This $\varphi(t)$ can be found since the number of zeros is finite and $\bar{F}$ is infinite. Then let $\varphi(a) = a$ for $a \in F$ and extend φ to the ring homomorphism φ: $F[t, y_1, \ldots, y_r] \to \bar{F}[y, \ldots, y_r]$, ($t, y_1, \ldots, y_r$ are algebraically independent over F).

Note that $\{y_1, \ldots, y_r\}$ is a base of M' since $e_1, \ldots, e_n$ are algebraic over $F(t)[y_1, \ldots, y_r]$ and the transcendence degree of $\{e_1, \ldots, e_n\}$ over $F(t)$ is r.

By the place extension theorem [**1**, Chapter I, Theorem 1] we may extend φ to a place $\varphi\colon \bar{K} \to \bar{K} \cup \{\infty\}$. Since $e_i \in E$ is integral over $F(t)[y_1, \ldots, y_r]$ it follows that $\varphi(e_i)$ is finite [**1**, Chapter I, Proposition 4]. We shall prove that the restriction of φ to E gives an algebraic representation of M over $\bar{F}$. The existence of an algebraic representation over F follows then by the lemma.

Now let $P_C^\varphi(X_C) \in \bar{F}(X_C)$ be the polynomial obtained by applying φ to the coefficients of P_C. Note that the polynomial P_C^φ is nonzero by the choice of $\varphi(t)$. Since $P_C^\varphi(\varphi(C)) = 0$ it follows that dependent subsets $A \subseteq E$ have images $\varphi(A)$ which are algebraically dependent over $\bar{F}$.

Then let B be a base of M. We shall prove that $\varphi(B)$ has transcendence degree r over $\bar{F}$. Let C_i be the fundamental circuit of y_i with respect to the base B and let $P_{C_i}(X_{C_i})$ be the associated polynomial. Since $P_{C_i}^\varphi(X_{C_i})$ contains all variables explicitly and $P_{C_i}^\varphi(\varphi(C_i)) = 0$, it follows that y_i $(= \varphi(y_i))$ is algebraic over $\bar{F}(\varphi(B))$. Since $y_1, \ldots, y_r$ the algebraically independent over F, we conclude that $\varphi(B)$ has transcendence degree r over $\bar{F}$.

Therefore $\varphi\colon E \to \bar{K}$ is an algebraic representation of M over the field generated by the coefficients of the polynomials P_C^φ, a field which is finitely generated algebraic over F. Repeated application of the lemma gives then an algebraic representation over F, and the theorem is proved. □

COROLLARY. *If a matroid M is algebraic over a field F, then any minor of M is algebraic over F.*

PROOF. By [**3**, Theorem 11.3.1] a minor of M is algebraic over some transcendental extension of F. The result follows then by our theorem. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF STOCKHOLM, S-113 85 STOCKHOLM, SWEDEN