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# INTERLACING POLYNOMIALS

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ABSTRACT. Let $A$ be an $n$-by-$n$ Hermitian matrix. We note that the set of all monic, degree $n-1$ polynomials whose roots interlace the eigenvalues of $A$ is exactly the classical field of values of $\operatorname{adj}(\lambda I - A)$.

Let $p$ be a polynomial of degree $n$ with *real* roots $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If $q$ is a degree $n-1$ polynomial with real roots $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$, we say that the roots of $q$ *interlace* those of $p$ if

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n,$$

and, in this event, we call $q$ an *interlacing polynomial* for $p$. We denote by $\operatorname{Int}(p)$ the set of all monic degree $n-1$ interlacing polynomials for a given degree $n$ polynomial $p$ with real roots.

OBSERVATION. *The set* $\operatorname{Int}(p)$ *is convex.*

PROOF. If $\lambda$ is a root of $p$ of multiplicity $k$, then $\lambda$ must be a root of multiplicity at least $k-1$ of each $q \in \operatorname{Int}(p)$. So, we assume, without loss of generality, that the roots of $p$ are

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

In this event, for $1 \leq i \leq n-1$, either

$$q(\lambda_i) \geq 0 \text{ and } q(\lambda_{i+1}) \leq 0 \quad \text{for all } q \in \operatorname{Int}(p)$$

or

$$q(\lambda_i) \leq 0 \text{ and } q(\lambda_{i+1}) \geq 0 \quad \text{for all } q \in \operatorname{Int}(p).$$

In either event, any convex combination of two polynomials in $\operatorname{Int}(p)$ must be monic and have a zero in the interval $[\lambda_i, \lambda_{i+1}]$, and thus lie in $\operatorname{Int}(p)$. $\square$

For an $n$-by-$n$ matrix $A$, the (classical) *field of values* of $A$ is defined (see [**2**] for a survey) by

$$F(A) = \{x^*Ax\colon x^*x = 1, x \in C^n\}.$$

Note that if the entries of $A$ are polynomials over $C$ in $\lambda$, then the elements of $F(A)$ are also polynomials in $\lambda$. Also let $A_i$ denote the $(n-1)$-by-$(n-1)$ principal submatrix of $A$ resulting from deletion of row and column $i$. It is well known that $F(A)$ is convex and that $F(A_i) \subseteq F(A)$ for a complex matrix $A$. Finally for a square matrix $A$, let $p_A(\lambda) = \det(\lambda I - A)$, the characteristic polynomial of $A$.

If $A$ is an $n$-by-$n$ Hermitian matrix, it is well known, e.g. [**1**], that each $p_{A_i} \in \operatorname{Int}(p_A)$, $i = 1, \ldots, n$. It is equally well known that $(1/n)p'_A \in \operatorname{Int}(p_A)$, where $p'_A$

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denotes the derivative of $p_A$. (Since $p'_A = \sum_{i=1}^n p_{A_i}$, the latter fact follows from the former and the convexity of $\mathrm{Int}(p_A)$.)

For an $n$-by-$n$ matrix $A$, let $A(\lambda) = \mathrm{adj}(\lambda I - A)$, so that the diagonal entries of $A(\lambda)$ are polynomials of degree $n-1$ in $\lambda$, the $p_{A_i}(\lambda)$, and the off-diagonal entries are polynomials of degree at most $n-2$ in $\lambda$. Note that, for $e_i$ the $i$th standard basis vector, $e_i^T A(\lambda) e_i = p_{A_i}(\lambda) \in \mathrm{Int}(p_A)$ if $A$ is Hermitian. It has also been noted [**3**] in connection with graph theory that $(1/n)e^T A(\lambda) e \in \mathrm{Int}(p_A)$, for $e = (1, \ldots, 1)^T$, and it can be shown that there is an $x \in C^n$, $x^*x = 1$, such that

$$x^* A(\lambda) x = (1/n)\mathrm{Tr}(A(\lambda)) = (1/n)p'_A(\lambda) \in \mathrm{Int}(p_A).$$

In each case an element of $F(A(\lambda))$ is a polynomial in $\mathrm{Int}(p_A)$ for $A$ Hermitian. Note that $(1/x^*x)x^*A(\lambda)x$ is a monic polynomial of degree $n-1$ in $\lambda$ in general for $0 \neq x \in C^n$.

In order to characterize the interlacing polynomials for a given Hermitian matrix and note that they all arise in essentially the same way, our goal is to make the following observation.

THEOREM. *If $A$ is an $n$-by-$n$ Hermitian matrix, then*

$$\mathrm{Int}(p_A(\lambda)) = F(\mathrm{adj}(\lambda I - A)).$$

PROOF. We first make some simple observations which are easily checked for any $n$-by-$n$ complex matrix $A$.

(1) $$p_{S^{-1}AS}(\lambda) = p_A(\lambda), \qquad S \text{ nonsingular};$$

(2) $$\mathrm{adj}(\lambda I - S^{-1}AS) = S^{-1}\,\mathrm{adj}(\lambda I - A)S, \qquad S \text{ nonsingular};$$

(3) $$F(U^*\,\mathrm{adj}(\lambda I - A)U) = F(\mathrm{adj}(\lambda I - A)), \qquad U \text{ unitary};$$

and

(4) each polynomial in $F(\mathrm{adj}(\lambda I - A))$ is monic of degree $n-1$.

Now, suppose $A$ is $n$-by-$n$ Hermitian with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and let $U$ be unitary so that

$$U^*AU = D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

By (2) and (3) $F(A(\lambda)) = F(D(\lambda))$, and by (1), $\mathrm{Int}(p_A) = \mathrm{Int}(p_D)$. So, we may assume, without loss of generality throughout, that $A$ is diagonal (i.e. $A = D$). In this event

(5) $$A(\lambda) = \mathrm{diag}\left(\prod_{j\neq 1}(\lambda - \lambda_j), \ldots, \prod_{j\neq n}(\lambda - \lambda_j)\right).$$

It is obvious that

(6) $$\text{each polynomial } \prod_{j\neq i}(\lambda - \lambda_j) \in \mathrm{Int}(p_A), \qquad i = 1, \ldots, n.$$

To verify that $F(\mathrm{adj}(\lambda I - A)) \subseteq \mathrm{Int}(p_A)$, let $x \in C^n$ with $x^*x = 1$. Then

$$x^*(\mathrm{adj}(\lambda I - A))x = \sum_{i=1}^n \bar{x}_i x_i \prod_{j\neq i}(\lambda - \lambda_j) \in \mathrm{Int}(p_A)$$

because of (5) and (6) and the fact that $\mathrm{Int}(p_A)$ is convex.

For the inclusion $\mathrm{Int}(p_A) \subseteq F(\mathrm{adj}(\lambda I - A))$, we show by induction on $n$ that every set of roots which interlace the eigenvalues of $A$ occurs as the roots of a polynomial in $F(A(\lambda))$. This is easily checked for $n = 2$; suppose it has been verified through $n-1$. Suppose that $\mu_1 \leq \cdots \leq \mu_{n-1}$ such that

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$$

are given. By the induction hypothesis there is an $x \in C^n$, with $x_{n-1} = 0$ and $x^*x = 1$, such that the degree $n-1$ monic polynomial $p_x(\lambda) = x^*A(\lambda)x$ has $\mu_1, \ldots, \mu_{n-2}$ among its roots and there is a $y \in C^n$, with $y_n = 0$ and $y^*y = 1$, such that the degree $n-1$ monic polynomial $p_y(\lambda) = y^*A(\lambda)y$ also has $\mu_1, \ldots, \mu_{n-2}$ among its roots. Note that the remaining root of $p_x$ is $\lambda_{n-1}$ and of $p_y$ is $\lambda_n$ and that

$$p_x(\lambda) = \sum_{i=1}^{n} \bar{x}_i x_i \prod_{j \neq i} (\lambda - \lambda_j)$$

while

$$p_y(\lambda) = \sum_{i=1}^{n} \bar{y}_i y_i \prod_{j \neq i} (\lambda - \lambda_j).$$

Let $p_\alpha = (1-\alpha)p_x + \alpha p_y$, so that $p_0 = p_x$ and $p_1 = p_y$, while $\mu_1, \ldots, \mu_{n-2}$ are roots of the degree $n-1$ monic polynomial $p_\alpha$ for all scalars $\alpha$. As $\alpha$ varies from 0 to 1, the remaining root of $p_\alpha$ varies continuously from $\lambda_{n-1}$ to $\lambda_n$. Let $\alpha_0$, $0 \leq \alpha_0 \leq 1$, be such that $p_{\alpha_0}(\mu_{n-1}) = 0$ and let

$$w_i = [(1-\alpha_0)\bar{x}_i x_i + \alpha_0 \bar{y}_i y_i]^{1/2}, \qquad i = 1, \ldots, n,$$

and $z_i = w_i/(w^*w)^{1/2}$. Then $p_z(\lambda) = z^*A(\lambda)z \in F(A(\lambda))$ is the element of $\mathrm{Int}(p_A)$ with roots $\mu_1, \ldots, \mu_{n-1}$. $\square$

REMARK. Since $p_{A_i}(\lambda) = e_i^T \,\mathrm{adj}(\lambda I - A)e_i \in F(A(\lambda))$, the classical interlacing inequalities for a Hermitian matrix follow from the theorem. As the classical interlacing inequalities were not used in the proof, a simple and self-contained demonstration of them is contained therein.

The sufficiency of the interlacing inequalities for a principal submatrix of a Hermitian matrix also follows from the theorem, as any element of $F(\mathrm{adj}(\lambda I - A))$ appears on the diagonal of an appropriately chosen unitary similarity.

REMARK. It follows from the theorem that for an $n$-by-$n$ Hermitian matrix $A$, $F(\mathrm{adj}(\lambda I - A))$ is convex, generalizing slightly the familiar fact that $F(B)$ is convex for an $n$-by-$n$ complex matrix $B$. If $B(\lambda)$ is a general polynomial matrix, is $F(B(\lambda))$ convex?

REMARK. It also follows from the above observations that $\mathrm{Int}(p)$ is the convex hull of

$$\left\{ \prod_{j \neq 1} (\lambda - \lambda_j), \ldots, \prod_{j \neq n} (\lambda - \lambda_j) \right\},$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the roots of $p$.

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