Abstract. We show that the algebra of double multipliers of a certain Banach algebra $A$ can be embedded in the second conjugate space $A^{**}$ of $A$. This generalizes the previous work by the author for $B^*$-algebras.

1. Introduction. Let $A$ be a semisimple Banach algebra and $M(A)$ the algebra of double multipliers on $A$. The author recently showed that $M(A)$ is isomorphic to $(A^{**}, \circ)$ if and only if $A$ has the following properties: (1) $A$ is Arens regular, (2) $A$ has an approximate identity, and (3) $\pi(A)$ is an ideal of $(A^{**}, \circ)$ [10, Theorem, p. 442]. In general, not many Banach algebras possess all these properties. The purpose of this paper is to establish an embedding of $M(A)$ in the second conjugate space $A^{**}$ of $A$ for a semisimple Banach algebra $A$ with an approximate identity. In fact, we show that $M(A)$ is isometrically isomorphic to $M^{**}/N^{**}$, where $M^{**} = \{F \in A^{**} : F \circ \pi(x) \text{ and } \pi(x) \circ F \in \pi(A) \text{ for all } x \in A\}$ and $N^{**} = \{F \in A^{**} : F(f \circ \pi) = F(x \circ' f) = 0 \text{ for all } x \in A \text{ and } f \in a^*\}$. The main idea of the proof of this result is essentially contained in the proofs of [8, Lemma 2.1, p. 80 and 10, Theorem, p. 422]. This embedding was studied in [1 and 8] for $B^*$-algebras.

2. Notation and preliminaries. Definitions not explicitly given are taken from Rickart’s book [6].

Let $A$ be a Banach algebra. Then $A^*$ and $A^{**}$ will denote the first and second conjugate spaces of $A$, and $\pi$ the canonical map of $A$ into $A^{**}$. The two Arens products on $A^{**}$ are defined in stages according to the following rules (see [2]). Let $x, y \in A$, $f \in A^*$, and $F, G \in A^{**}$.

Define $f \circ x$ by $(f \circ x)(y) = f(xy)$. Then $f \circ x \in A^*$.

Define $G \circ f$ by $(G \circ f)(x) = G(f \circ x)$. Then $G \circ f \in A^*$.

Define $F \circ G$ by $(F \circ G)(f) = F(G \circ f)$. Then $F \circ G \in A^{**}$.

Define $x \circ' f$ by $(x \circ' f)(y) = f(yx)$. Then $x \circ' f \in A^*$.

Define $f \circ' F$ by $(f \circ' F)(x) = F(x \circ' f)$. Then $f \circ' F \in A^*$.

Define $F \circ' G$ by $(F \circ' G)(f) = G(f \circ' F)$. Then $F \circ' G \in A^{**}$.

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$A^{**}$ is a Banach algebra under the products $F \circ G$ and $F \circ' G$ and $\pi$ is an algebra isomorphism of $A$ into $(A^{**}, \circ)$ and $(A^{**}, \circ')$. In general, $\circ$ and $\circ'$ are distinct on $A^{**}$. If they agree on $A^{**}$, then $A$ is called Arens regular.

The following properties of the Arens products will be used often in the rest of this paper.

**Lemma 2.1.** Let $A$ be a Banach algebra. Then for all $x \in A$, $f \in \mathcal{A}^*$, and $F, G \in A^{**}$, we have

1. $\pi(x) \circ F = \pi(x) \circ' F$ and $F \circ \pi(x) = F \circ' \pi(x)$.
2. If $\{F_\beta\} \subset A^{**}$ and $F_\beta \to F$ weakly in $A^{**}$, then $F_\beta \circ G \to F \circ G$ and $G \circ' F_\beta \to G \circ' F$ weakly.
3. $F(x \circ' f) = (F \circ \pi(x))(f)$.

**Proof.** (1) is proved in [2, p. 843] and (2) in [2, p. 842].

(3) It is easy to see that $\pi(x) \circ f = x \circ' f$. Therefore

\[
(F \circ \pi(x))(f) = F(\pi(x) \circ f) = F(x \circ' f).
\]

This completes the proof of the lemma.

We say that a Banach algebra $A$ has an approximate identity if there exists a net $(e_\alpha)$ in $A$ such that $\|e_\alpha\| \leq 1$ for all $\alpha$ and $x = \lim e_\alpha x = \lim x e_\alpha$ for all $x \in A$.

Let $A$ be a semisimple Banach algebra. A pair $(T_1, T_2)$ of operators from $A$ to $A$ is called a double multiplier (centralizer) on $A$ provided that $x(T_1 y) = (T_2 x) y$ for all $x, y \in A$. It is knowns that $T_1$ and $T_2$ are continuous linear operators on $A$ such that $T_1(xy) = (T_1 x)y$ and $T_2(xy) = x(T_2 y)$. The set $M(A)$ of all double multipliers on $A$ is a Banach algebra with identity and $A$ can be identified as a two-sided ideal of $M(A)$ (see [4 and 5]).

In this paper, all algebras and spaces under consideration are over the complex field.

**3. Extensions of double multipliers.** Let $A$ be a semisimple Banach algebra. We show that each double multiplier $T = (T_1, T_2)$ on $A$ can be extended to a double multiplier on $(A^{**}, \circ)$. In fact, for all $x \in A$ and $f \in \mathcal{A}^*$, we define

\[
(f \circ T_1)(x) = f(T_1 x) \quad \text{and} \quad (f \circ T_2)(x) = f(T_2 x).
\]

Then $f \circ T_1$ and $f \circ T_2 \in \mathcal{A}^*$. For all $F \in A^{**}$, we define

\[
T_1^*(F)(f) = F(f \circ T_1) \quad \text{and} \quad T_2^*(F)(f) = F(f \circ T_2).
\]

Then $T_1^*(f)$ and $T_2^*(F) \in A^{**}$.

**Theorem 3.1.** Let $A$ be a semisimple Banach algebra. Then for each $T = (T_1, T_2) \in M(A)$, $T^* = (T_1^*, T_2^*)$ is a continuous double multiplier on $(A^{**}, \circ)$ and $T^* | \pi(A) = T$.  

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Proof. Let $F$ and $G \in A^{**}$. Then by Goldstine’s Theorem, there exist nets $\{x_m\}$ and $\{y_n\}$ in $A$ such that $\pi(x_m) \to F$ and $\pi(y_n) \to G$ weakly in $A^{**}$. Then for all $f \in A^*$, we have

\[
(F \circ T_1^*(G))(f) = \lim_m \left( T_1^*(G) \circ f \right)(x_m) = \lim_m G\left( (f \circ x_m) \circ T_1 \right)
\]

\[
= \lim_m \lim_n \left( (f \circ x_m) \circ T_1 \right)(y_n)
\]

\[
= \lim_m \lim_n f\left( (T_2 x_m) y_n \right) = \lim_m G\left( f \circ T_2 x_m \right)
\]

\[
= \lim_m \left( (G \circ f) \circ T_2 \right)(x_m) = \left( T_2^*(F) \circ G \right)(f).
\]

Therefore $F \circ T_1^*(G) = T_2^*(F) \circ G$. Similarly, we can show that $T_1^*(F \circ G) = T_1^*(F) \circ G$ and $T_2^*(F \circ G) = F \circ T_2^*(G)$. It is easy to see that $\|T_1^*\| \leq \|T_1\|$ and $\|T_2^*\| \leq \|T_2\|$. Therefore $\|T^*\| \leq \|T\|$. Since $T_1^*(\pi(x)) = \pi(T_1 x)$ and $T_2^*(\pi(x)) = \pi(T_2 x)$, we have $T^* | \pi(A) = T$ and the theorem is proved.

4. The subalgebras $M^{**}$ and $N^{**}$ of $(A^{**}, \circ)$. For any Banach algebra $A$, set

\[
M^{**} = \{ F \in A^{**} : F \circ \pi(x) \text{ and } \pi(x) \circ F \in \pi(A) \text{ for all } x \in A \}
\]

and

\[
N^{**} = \{ F \in A^{**} : F(\circ x) = F(x \circ f) = 0 \text{ for all } x \in A \text{ and } f \in A^* \}.
\]

Then $\pi(A)$ is a two-sided ideal of $(A^{**}, \circ)$ if and only if $M^{**} = A^{**}$.

Lemma 4.1. Let $A$ be a Banach algebra. Then $M^{**}$ and $N^{**}$ are closed subalgebras of $(A^{**}, \circ)$ and $(A^{**}, \circ')$. Also $N^{**}$ is a two-sided ideal in $M^{**}$ and $N^{**} = \{ F \in A^{**} : A^{**} \circ F = F \circ A^{**} = (0) \}$.

Proof. For any $F \in N^{**}$, easy calculations show that $\pi(A) \circ F = (0)$ and $F \circ \pi(A) = (0)$. Hence $A^{**} \circ F = F \circ A^{**} = (0)$ and so $N^{**} = \{ F \in A^{**} : A^{**} \circ F = F \circ A^{**} = (0) \}$. Thus if $F, G \in N^{**}$, then $\pi(A) \circ (F \circ G) = (F \circ G) \circ \pi(A) = (0)$ and so $F \circ G \in N^{**}$. Similarly, $F \circ' G \in N^{**}$. It is easy to check that $M^{**}$ and $N^{**}$ are closed subalgebras in $(A^{**}, \circ)$ and $(A^{**}, \circ')$ and $N^{**}$ is a two-sided ideal of $M^{**}$. Therefore the lemma is proved.

In the rest of this section, $A$ will be a semisimple Banach algebra with an approximate identity $\{e_\alpha\}$. It follows from [3, Proposition 7, p. 146] that $(A^{**}, \circ)$ has a right identity $I$. Also $I$ is a left identity for $(A^{**}, \circ')$.

Theorem 4.2. Let $A$ be a semisimple Banach algebra with an approximate identity $\{e_\alpha\}$. Then the Arens products $\circ$ and $\circ'$ agree on $M^{**}/N^{**}$ and $M^{**}/N^{**}$ is a Banach algebra with an identity.

Proof. Let $F, G \in M^{**}$, $f \in A^*$, and $x \in A$. Then by Cohen’s Factorization Theorem [3, Theorem 10, p. 61], $x = yz$ with $y, z \in A$. Hence

\[
(F \circ G - F \circ' G)(f \circ x) = (F \circ G - F \circ' G)((f \circ y) \circ z)
\]

\[
= (\pi(z) \circ F \circ G - \pi(z) \circ F \circ G)(f \circ y) = 0.
\]
Similarly, \((F \circ G - F \circ' G)(x \circ' f) = 0\). Therefore \(F \circ G - F \circ' G \in N^{**}\) and so \(\circ\) and \(\circ'\) agree on \(M^{**}/N^{**}\). Since \(I\) is a right identity for \((A^{**}, \circ)\) and a left identity for \((A^{**}, \circ)\) and \(I \in M^{**}\), it follows that \(M^{**}/N^{**}\) is a Banach algebra with an identity.

**Remark.** For a certain Banach algebra \(A\), \(N^{**}\) is the radical of \((A^{**}, \circ)\) (for example, see [9, Theorem 4.1, p. 440]).

The following result is essentially contained in the proof of [8, Lemma 2.1, p. 80]. It is useful in the next section.

**Lemma 4.3.** Let \(A\) be a semisimple Banach algebra with an approximate identity \(\{e_a\}\), \(T = (T_1, T_2) \in M(A)\), and \(x \in A\). Then
1. \(T_1^{**}(I) \circ \pi(x) = T_2^{**}(I) \circ \pi(x) = \pi(T_1x)\).
2. \(\pi(x) \circ T_1^{**}(I) = \pi(x) \circ T_2^{**}(I) = \pi(T_2x)\).
3. \(T_1^{**}(I) = T_2^{**}(I) + N\) for some \(N \in N^{**}\).
4. If \(T_1^{**}(I)\) and \(T_2^{**}(I)\) are both in \(N^{**}\), then \(T = 0\).

**Proof.** (1) For all \(f \in A^{*}\), we have
\[
(T_1^{**}(I) \circ \pi(x))(f) = I((\pi(x) \circ f) \circ T_1) = \lim_{\alpha} ((\pi(x) \circ f) \circ T_1)(e_{\alpha}) = \lim_{\alpha} f(T_1(e_{\alpha}x)) = \lim_{\alpha} (f \circ T_1)(e_{\alpha}x) = (f \circ T_1)(x) = f(T_1x).
\]
Hence \(T_1^{**}(I) \circ \pi(x) = \pi(T_1x)\). Similarly, \(T_2^{**}(I) \circ \pi(x) = \pi(T_2x)\). Therefore (1) is proved. Similarly we can prove (2).

(3) Let \(N = T_1^{**}(I) - T_2^{**}(I)\). Then
\[
N(f \circ x) = \pi(x)(N \circ f) = (\pi(x) \circ N)(f) = (\pi(x) \circ T_1^{**}(I) - \pi(x) \circ T_2^{**}(I))(f) = (\pi(T_2x) - \pi(T_2x))(f) = 0.
\]
Similarly, \(N(x \circ' f) = 0\). Therefore \(N \in N^{**}\) and so \(T_1^{**}(I) = T_2^{**}(I) + N\).

(4) Since \(T_1^{**}(I)\) and \(T_2^{**}(I) \in N^{**}\), we have
\[
\pi(T_1x)(f) = (T_2^{**}(I) \circ \pi(x))(f) = T_2^{**}(I)(x \circ' f) = 0.
\]
Hence \(T_1x = 0\) and so \(T_1 = 0\). Also
\[
\pi(T_2x)(f) = (\pi(x) \circ T_1^{**}(I))(f) = T_1^{**}(I)(f \circ x) = 0.
\]
Therefore \(T_2 = 0\) and so \(T = 0\). This completes the proof of the lemma.

**Notation.** In the rest of this paper, for each \(F \in M^{**}\), we write \(\tilde{F} = F + N^{**}\).

**5. The algebras** \(M(A)\) **and** \(M^{**}/N^{**}\). **We have the main result of this paper.**

**Theorem 5.1.** Let \(A\) be a semisimple Banach algebra with an approximate identity \(\{e_a\}\). Then \(M(A)\) is isometrically isomorphic to \(M^{**}/N^{**}\).

**Proof.** Let \(T = (T_1, T_2) \in M(A)\). Then by Lemma 4.3, \(T_1^{**}(I)\) and \(T_2^{**}(I) \in M^{**}\) and \(T_1^{**}(I) = T_2^{**}(I)\). We define a mapping \(\Phi\) from \(M(A)\) to \(M^{**}/N^{**}\) by
\[
\Phi(T) = T_1^{**}(I) = T_2^{**}(I).
\]
We show that \( \Phi \) is an isometric isomorphism from \( M(A) \) onto \( M^{**}/N^{**} \). It is clear that \( \Phi \) is linear. Let \( x \in A \) and \( f \in A^* \). If \( \Phi(T) = 0 \), then \( T_1^\#(I) \) and \( T_2^\#(I) \) are both in \( N^{**} \). Hence by Lemma 4.3, \( T = 0 \). Therefore \( \Phi \) is one-one. Let \( S = (S_1, S_2) \in M(A) \). Since \( ST = (S_1T_1, T_2S_2) \), by Lemma 4.3, we have

\[
(S_1^\#(I) \circ T_1^\#(I))(f \circ x) = (\pi(x) \circ S_1^\#(I) \circ T_1^\#(I))(f) = (\pi(S_2x) \circ T_2^\#(I))(f) = \pi(T_2S_2x)(f) = (\pi(x) \circ S_1^\#(I))^\#(I)(f \circ x).
\]

Similarly, we have

\[
(S_1^\#(I) \circ T_1^\#(I))(x \circ f) = (S_1T_1)^\#(I)(x \circ f).
\]

Therefore \( \Phi(ST) = \Phi(S)\Phi(I) \) and so \( \Phi \) is an isomorphism. Let \( F \in M^{**} \). Define

\[
P_1(x) = F \circ \pi(x) \quad \text{and} \quad P_2(x) = \pi(x) \circ F \quad (x \in A).
\]

Since \( F \in M^{**} \), \( P = (P_1, P_2) \in M(A) \). Also

\[
P_1^\#(I)(f \circ x) = I((f \circ x) \circ P_1) = \lim \alpha ((f \circ x) \circ P_1)(e_\alpha)
\]

\[
= \lim \alpha f(x(F \circ \pi(e_\alpha))) = \lim \alpha f(\pi(x) \circ F)e_\alpha
\]

\[
= \lim \alpha f(e_\alpha(F \circ x)) = \lim \alpha f(\pi(e_\alpha x) \circ F)
\]

\[
= \lim \alpha (\pi(e_\alpha x) \circ F)(f) = \lim \alpha \pi(e_\alpha x)(F \circ f)
\]

\[
= (F \circ f)(x) = F(f \circ x).
\]

Similarly, we have \( P_2^\#(I)(x \circ f) = F(x \circ f) \). Therefore \( \Phi(P) = \tilde{P}^\# = \tilde{F} \). Hence \( \Phi \) is an onto mapping. It is easy to see that \( ||\Phi(T)|| < ||T|| \). Since \( ||T_1|| = \sup\{|f(T_1x)|: \|x\| \leq 1, \|f\| \leq 1 \text{ for all } x \in A \text{ and } f \in A^* \} \), for given \( \epsilon > 0 \), there exists \( y \in A \) and \( g \in A^* \) with \( \|y\| \leq 1 \) and \( \|g\| \leq 1 \) such that \( |g(T_1y)| > ||T_1|| - \epsilon \). Then

\[
I((y \circ g) \circ T_1) = \lim \alpha (y \circ g)(T_1e_\alpha) = \lim \alpha g((T_1e_\alpha)y) = \lim g(T_1(e_\alpha y))
\]

\[
= \lim \alpha (g \circ T_1)(e_\alpha y) = (g \circ T_1)(y) = g(T_1y).
\]

Hence for all \( N \in N^{**} \), we have

\[
||T_1^\#(I) + N|| = |(T_1^\#(I) + N)(y \circ g)| = |T_1^\#(I)(y \circ g)| = |I((y \circ g) \circ T_1)| = |g(T_1y)| > ||T_1|| - \epsilon.
\]

Since \( \epsilon \) is arbitrary, we have

\[
||\Phi(T)|| = ||\tilde{T}_1^\#(I)|| > ||T_1||.
\]

Similarly, we can show that \( ||\Phi(T)|| > ||T_2|| \) and so \( ||\Phi(T)|| > ||T|| \). Therefore \( ||\Phi(T)|| = ||T|| \). This completes the proof of the theorem.

**Corollary 5.2.** Let \( A \) be as in Theorem 5.1. If \( (A^{**}, \circ) \) has an identity \( I \), then \( N^{**} = \{0\} \) and so \( M(A) \) is isometrically isomorphic to \( M^{**} \).
Proof. Let $N \in N^{**}$. Then for all $f \in A^*$, we have

$$N(f) = (I \circ N)(f) = \lim_{\alpha} (\pi(e_\alpha) \circ N)(f) = \lim_{\alpha} N(f \circ e_\alpha) = 0.$$ 

Therefore $N^{**} = (0)$ and the corollary follows from Theorem 5.1.

**Corollary 5.3.** Let $A$ be as in Theorem 5.1. If $A$ is Arens regular, then $M(A)$ is isometrically isomorphic to $M^{**}$.

**Proof.** Since $A$ is Arens regular, $(A^{**}, \circ)$ has an identity $I$. Therefore the result follows from Corollary 5.2.

We now have a slight improvement of [10, Theorem, p. 442].

**Theorem 5.4.** Let $A$ be a semisimple Banach algebra. Then $M(A)$ is isometrically isomorphic to $(A^{**}, \circ)$ if and only if $A$ has the following properties:

1. $A$ is Arens regular.
2. $A$ has an approximate identity.
3. $\pi(A)$ is a two-sided ideal of $(A^{**}, \circ)$.

**Proof.** Suppose that $M(A)$ is isometrically isomorphic to $(A^{**}, \circ)$. Then by Theorem 5.1, we have $N^{**} = (0)$ and $M^{**} = A^{**}$. Hence $\pi(A)$ is a two-sided ideal of $(A^{**}, \circ)$. By Theorem 4.2, $A$ is Arens regular and $(A^{**}, \circ)$ has an identity. Hence it follows from [3, Proposition 7, p. 147] that $A$ has an approximate identity. Therefore $A$ has properties (1), (2), and (3).

Conversely, suppose that $A$ has properties (1), (2), and (3). Then $M^{**} = A^{**}$ and so by Corollary 5.3, $M(A)$ is isometrically isomorphic to $(A^{**}, \circ)$. This completes the proof of the theorem.

**6. Banach *-algebras.** Let $A$ be a Banach *-algebra with a continuous involution. For all $x \in A$, $f \in A^*$, and $F \in A^{**}$, we define

$$f^*(x) = \overline{f(x^*)} \quad \text{and} \quad F^*(f) = \overline{F(f^*)},$$

where the bar denotes the complex conjugation. Then $f^* \in A$ and $F^* \in A^{**}$. If $A$ is a $B^*$-algebra, then $(A^{**}, \circ)$ is a $B^*$-algebra under the involution $F \to F^*$ (see [7, p. 192]).

**Lemma 6.1.** Let $A$ be a Banach *-algebra with a continuous involution. Then

1. For all $F$ and $G \in A^{**}$, we have

$$\left( F \circ G \right)^* = G^* \circ' F^* \quad \text{and} \quad \left( F \circ' G \right)^* = G^* \circ F^*.$$ 

2. $A$ is Arens regular if and only if $(A^{**}, \circ)$ is a *-algebra.

**Proof.** (1) For all $x \in A$ and $F \in A^*$, it is easy to show that $f^* \circ x = (x^* \circ' f)^*$ and so $G^* \circ f^* = (f^* \circ' G^*)^*$. Then

$$\left( F \circ G \right)^*(f) = \overline{F(G \circ f^*)} = \overline{F(\left( f^* \circ' G^* \right)^*)} = F^*(f^* \circ' G^*) = (G^* \circ' F^*)(f).$$

Hence $(F \circ G)^* = G^* \circ' F^*$. Similarly, we have $(F \circ' G)^* = G^* \circ F^*$.
(2) If $A$ is Arens regular, then it follows from (1) that $(A^{**}, \circ)$ is a $*$-algebra. Conversely, suppose that $(A^{**}, \circ)$ is a $*$-algebra. Then
\[ F \circ G = (F \circ G)^{**} = (G^* \circ F)^* = F^{**} \circ G^{**} = F \circ G. \]

Hence $A$ is Arens regular and the lemma is proved.

If $A$ is a semisimple Banach $*$-algebra and $T = (T_1, T_2) \in M(A)$, then $T = (T_1, T_2) \rightarrow T^* = (T_2^*, T_1^*)$ is a continuous involution on $M(A)$, where $T_i^*(x) = (T_i(x^*))^*$ (i = 1, 2).

**Theorem 6.2.** Let $A$ be a semisimple Banach $*$-algebra. If $A$ is Arens regular and $A$ has an approximate identity $\{e_a\}$ with $e_a^* = e_a$, then $M(A)$ is isometrically $*$-isomorphic to $M^{**}$.

**Proof.** By Corollary 5.3, $M(A)$ is isometrically isomorphic to $M^{**}$. By Lemma 6.1, $(A^{**}, \circ)$ is a $*$-algebra and so $M^{**}$ is a closed $*$-subalgebra of $(A^{**}, \circ)$. Let $T = (T_1, T_2) \in M(A)$. Then, for all $f \in A^*$, we have
\[
(T_1^*)^\#(I)(f) = \lim_{\alpha} f \circ T_1^*(e_\alpha) = \lim_{\alpha} f(T_1^* e_\alpha) = \lim_{\alpha} f(\overline{\overline{T_1^* e_\alpha}}) = I(f \circ T_1^*) = T_1^* \circ T_1^* (f).
\]

Hence $(T_1^*)^\#(I) = (T_1^\#(I))^*$. Therefore $\Phi(T^*) = (\Phi(T))^*$. This completes the proof of the theorem.

If $A$ is a $B^*$-algebra, then $A$ is Arens regular and $A$ has an approximate identity. Therefore we have the following result:

**Corollary 6.3.** If $A$ is a $B^*$-algebra, then $M(A)$ is isometrically $*$-isomorphic to $M^{**}$.

**Remark.** Corollary 6.3 is [8, Lemma 2.1, p. 80].

**References**


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