POLYNOMIALS OF AN INNER FUNCTION
WHICH ARE EXPOSED POINTS IN $H^1$

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Abstract. It is known that if $p(z)$ is an analytic polynomial which has no zeros in
the open unit disc and distinct zeros in the unit circle, then $p(z)/\|p(z)\|_1$ is an
exposed point of the unit ball of the Hardy space $H^1$.

In this paper, it is proved that for a bounded analytic function $f$ with $\|f\|_\infty \leq 1$, $p(f)/\|p(f)\|_1$ is also an exposed point.

Let $U$ be the open unit disc in the complex plane and let $\partial U$ be the boundary of
$U$. If $f$ is analytic in $U$ and $\int_{-\pi}^{\pi} \log|f(re^{i\theta})| \, d\theta$ is bounded for $0 \leq r < 1$, then
$f(e^{i\theta})$, which we define to be $\lim_{r \to 1} f(re^{i\theta})$, exists almost everywhere on $\partial U$. If
\[
\lim_{r \to 1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta = \int_{-\pi}^{\pi} \log^+ |f(e^{i\theta})| \, d\theta,
\]
then $f$ is said to be in the class $N_+$. The set of all boundary functions in $N_+$ is
denoted by $N_+$ again. For $0 < p \leq \infty$, the Hardy space $H^p$ is defined by $N_+ \cap L^p$.
$A$ denotes the disc algebra, that is $A = \{ f : f$ is continuous on $\overline{U}$ and analytic in $U \}$. If
$h$ in $N_+$ has the form
\[
h(z) = \exp \left( \int_{-\pi}^{\pi} e^{it} + z \log|h(e^{it})| \, dt + i\alpha \right)
\]
for some real $\alpha$, $h$ is called an outer function. We call $q$ in $N_+$ an inner function if
$|q(e^{i\theta})| = 1$ a.e. on $\partial U$.

Let $g$ be a nonzero function in $H^p$. Then the following property $(\ast)$ characterizes
that $g$ is an outer function.

$(\ast)$ Whenever $kg$ belongs to $H^p$ for $k$ in $L^\infty$ with $k(e^{i\theta}) \geq 0$ a.e. on $U$, then $k$ is
a constant function (see [6]).

We can consider a stronger property of $g$:

$(\ast\ast)$ Whenever $kg$ belongs to $H^p$ for some Lebesgue measurable $k$ with
$k(e^{i\theta}) \geq 0$ a.e. on $\partial U$, then $k$ is a constant function.

In [6], the function $g$ with property $(\ast\ast)$ is called a $p$-strong outer function. We
should remark that deLeeuw and Rudin [1] used the phrase “strong outer function”
in a little different context. The $p$-strong outer functions appear to be important in
many problems, for example, extremal problems, interpolation problems, Toeplitz
operators, and prediction theory. In particular, when $\|g\|_1 = 1$, $g$ is a 1-strong outer function if and only if $g$ is exposed points of the unit ball of $H^1$ (see [6]).

Suppose $p(z) = \prod_{j=1}^{n} (z + a_j)$. If $p(z)/\|p(z)\|_1$ is an exposed point, then $|a_j| \geq 1$ $(j = 1, \ldots, n)$ and $a_i \neq a_j$ $(i \neq j)$ (cf. [1]). It is known that the converse is valid [7], which is also derived from [4 and 5] as follows. If $n = 1$, the result follows from Proposition 5 of [4]. Suppose $n > 1$ and $\prod_{j=1}^{n-1} (z + a_j)/\|\prod_{j=1}^{n-1} (z + a_j)\|_1$ is exposed but $p(z)/\|p(z)\|_1$ is not. Here we may assume without loss of generality that $|a_j| = 1$ for some $j$, say $j = n$. By Proposition 1 of [4], we have an element $k$ in $S_{|p|/p}$ which can be represented by $(e^{i\theta} + a_n)(1 + \bar{a}_n e^{i\theta})h(e^{i\theta})$ for some nonconstant $h$ in $H^1$ by Lemma 3 of [5]. Thus we have

$$k/p = (1 + \bar{a}_n z)h/\prod_{j=1}^{n-1} (z + a_j) > 0 \text{ a.e. on } \partial U,$$

which contradicts the assumption that $\prod_{j=1}^{n-1} (z + a_j)/\|\prod_{j=1}^{n-1} (z + a_j)\|_1$ is exposed.

Now we wish to prove that $p(f)/\|p(f)\|_1$ is an exposed point for the above $p(z)$ and any nonconstant $f$ in $H^\infty$ with $\|f\|_\infty \leq 1$. For $n = 1$, this is known [4, Proposition 5]. But we need a new idea to prove it in general.

**Lemma.** If $P(z) = \prod_{j=1}^{n} (z + a_j)$, $\|a_j| = 1 \ (j = 1, \ldots, n)$, and $a_i \neq a_j$ $(i \neq j)$, then there exists a $k$ in $A$ such that $k^{-1}$ is in $A$ and $\text{Re}[k(e^{i\theta})p(e^{i\theta})] \geq 0$ a.e. on $\partial U$.

**Proof.** By the hypothesis on $a_j$, we can write $a_j = e^{i(\alpha_j - \pi)}$ $(j = 1, \ldots, n)$, where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 2\pi$. Let

$$s = \left[\sum_{j=1}^{n} (\alpha_j - \pi)\right] / 4\pi$$
and

$$\alpha = 2\pi \left(\sum_{j=1}^{n} (\alpha_j - \pi) / 4\pi - s\right),$$

where $[-]$ is the greatest integer function, and we have $0 \leq \alpha < 2\pi$. Then there exists a real valued function $\nu(\theta)$ on $[0, 2\pi]$ such that

(i) $\nu(\theta) = e^{i\nu(\theta)}$ $(0 < \theta < 2\pi, \theta \neq \alpha_j, j = 1, \ldots, n)$,

(ii) $\nu(0) = \alpha$, $\nu(2\pi) - \nu(0) = 2n\pi$,

(iii) $\nu(\theta)$ is right continuous, and left continuous except for jump discontinuities of $\pi$ at $\alpha_j$ ($j = 1, \ldots, n$).

Indeed, $\nu(\theta)$ has the form

$$\nu(\theta) = \begin{cases} \alpha + j\pi + n\theta/2 & \text{if } \alpha_j \leq \theta < \alpha_{j+1}, \ j = 0, 1, \ldots, n, \\ \alpha + 2n\pi & \text{if } \theta = 2\pi, \end{cases}$$

where $\alpha_0 = 0$ and $\alpha_{n+1} = 2\pi$. Then there exists a continuous function $\nu_0$ on $[0, 2\pi]$ such that

(i) $\nu_0(\alpha_j) = -\alpha + j\pi - (n/2)a_j$ $(j = 1, \ldots, n)$,

(ii) $\nu_0(0) = \nu_0(2\pi) = -\alpha$,

(iii) $\nu_0$ is a straight line in each interval $[\alpha_j, \alpha_{j+1}]$ $(j = 0, \ldots, n)$.

Now we can find the desired function $k$ of the lemma. Let $\nu_0^*$ be the harmonic conjugate of $\nu_0$, then $\nu_0 + iv_0^*$ belongs to $A$ because $\nu_0$ is in a Lipschitz class (cf. [3, p. 140]). Let $k = -i \exp(-\nu_0^* + iv_0)$; then both $k$ and $k^{-1}$ are in $A$ and

$$k(e^{i\theta})p(e^{i\theta}) / |k(e^{i\theta})p(e^{i\theta})| = e^{i(\nu(\theta) + \nu_0(\theta) - \pi/2)}$$

with $-\pi/2 \leq \nu(\theta) + \nu_0(\theta) - \pi/2 \leq -\pi/2$ $(0 \leq \theta \leq 2\pi)$. 

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**Theorem.** If \( p(z) = \prod_{j=1}^n (z + a_j) \), \( |a_j| \geq 1 \) (\( j = 1, \ldots, n \)), and \( a_i \neq a_j \) (\( i \neq j \)), then for any nonconstant function \( f \) in \( H^\infty \) with \( \|f\|_\infty \leq 1 \), \( p(f)/\|p(f)\|_1 \) is an exposed point of the unit ball of \( H^1 \).

**Proof.** Let \( \Omega_1 = \{ j \mid 1 \leq j \leq n, |a_j| = 1 \} \), \( \Omega_2 = \{ j \mid 1 \leq j \leq n, |a_j| > 1 \} \), and put \( p_i(z) = \prod_{j \in \Omega_i} (z + a_j) \), where \( p_i(z) = 1 \) if \( \Omega_i \) is empty (\( i = 1, 2 \)). By the lemma there exists a \( k \) in \( A \) such that \( k^{-1} \) is in \( A \) and \( \text{Re}[k(e^{i\theta})p_1(e^{i\theta})] \geq 0 \) on \( \partial U \). So, \( \text{Re}[k(e^{i\theta})p_1(e^{i\theta})] > 0 \) on \( U \) by the Poisson integral representation of \( h(z)p_1(z) \). For any nonconstant \( f \) in \( H^\infty \) with \( \|f\|_\infty \leq 1 \), \( k(f(z)) \) is bounded analytic in \( U \), and

\[
\text{Re}[k(f(z))p_2(f(z))^{-1}p(f(z))] = \text{Re}[k(f(z))p_1(f(z))] > 0
\]
on \( U \), and hence \( \geq 0 \) a.e. on \( \partial U \). Then, by Proposition 5(2) of [3], we have that \( p(f)/\|p(f)\|_1 \) an exposed point of \( H^1 \).

**References**


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