FOR EVERY CONTINUOUS \( f \)

THERE IS AN ABSOLUTELY CONTINUOUS \( g \)

SUCH THAT \([f = g]\) IS NOT BILATERALLY STRONGLY POROUS

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ABSTRACT. For any Darboux function \( f : [0,1] \rightarrow \mathbb{R} \) and any \( 0 < \delta < 1 \) there is a point \( x \in [0,1 - \delta] \) and a sequence \( x_n \) such that

(a) \( x_n \in [x + \delta^{n+1}, x + \delta^n] \) \((n = 1,2,\ldots)\) and
(b) \( \sum_{n=2}^{\infty} |f(x_n) - f(x_{n-1})| < +\infty. \)

Consequently, for every \( f \in C[0,1] \) there is an absolutely continuous function \( g \) such that \( \{x : f(x) = g(x)\} \) is not bilaterally strongly porous.

In [1] P. Humke and M. Laczkovich proved that every continuous function agrees with an absolutely continuous function on a set which is not bilaterally strongly \( x^{1+\delta} \) porous (Theorem 3).

In fact, they proved that for any \( f \in C[0,1] \) there is a point \( x \in [0,1] \) and a sequence \( x_n \rightarrow x \) such that

\[ x_n \in [x + ((n + 1)!)^{-\delta}, x + (n!)^{-1-\delta}] \]

and

\[ |f(x_n) - f(x)| < n^{-1-\delta/2} \]

for \( n > n_0 \) and \( \delta > 0 \).

Let \( g \in C[0,1] \) be linear on the intervals \([0,x], [x_{n+1}, x_n] \) \((n = 1,2,\ldots), [x_1, 1] \), and agree with \( f \) at the points \( x_n \). Since

\[ \sum_{n=n_0}^{\infty} |f(x_n) - f(x_{n-1})| < +\infty, \]

g is absolutely continuous and \( f = g \) on a set which is not bilaterally strongly \( x^{1+\delta} \)-porous. We prove the following

THEOREM. For any Darboux function \( f : [0,1] \rightarrow \mathbb{R} \) and any \( 0 < \delta < 1 \) there is a point \( x \in [0,1 - \delta] \) and a sequence \( x_n \) such that

(a) \( x_n \in [x + \delta^{n+1}, x + \delta^n] \) \((n = 1,2,\ldots)\) and
(b) \( \sum_{n=2}^{\infty} |f(x_n) - f(x_{n-1})| < +\infty. \)

Defining the function \( g \) as above, we obtain the following

COROLLARY. For every \( f \in C[0,1] \) there is an absolutely continuous function \( g \) such that \( \{x : f(x) = g(x)\} \) is not bilaterally strongly porous.

DEFINITION 1. Let \( R \) be a rectangle \( R = [\alpha, \beta] \times [m, M] \). We say that \( f \) is in \( R \) if \( \text{graf } f|_{[\alpha, \beta]} \subset R \).

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From now on we suppose that we are given a positive number $\delta < 1$ and a Darboux function $f: [0, 1] \to \mathbb{R}$.

**Definition 2.** Suppose that we have a rectangle $R = [\alpha, \beta] \times [m, M]$ and a number $k \in \mathbb{N}$ such that $\delta^k < \beta - \alpha$ and $f$ is in $R$. We say that $f$ is $k$-first class in $R$ if whenever $I \subseteq [\alpha + \delta^{k+1}, \beta]$ with $|I| = \delta^k (1 - \delta)$, then there exists $x \in I$ such that $f(x) = (m + M)/2$. Thus we can define a function $y_k : [\alpha, \beta - \delta^k] \to \mathbb{R}$ such that for any $x \in [\alpha, \beta - \delta^k]$ we have $f(y_k(x)) = (m + M)/2$ and $y_k(x) \in [x + \delta^{k+1}, x + \delta^k]$. We say that $f$ is $k$-second class in $R$ if $f$ is not $k$-first class in $R$; that is if there is an $x' \in [\alpha, \beta - \delta^k]$ such that $f(x) \neq (m + M)/2$ for $x \in [x' + \delta^{k+1}, x' + \delta^k]$.

We choose a number $p \in \mathbb{N}$ such that $(1 - \delta^p) - \sum_{n=p}^{\infty} \delta^n \geq \delta$.

**Lemma.** Suppose that $k \in \mathbb{N} \cup \{0\}$, $m < M \in \mathbb{R}$, $\alpha + \delta^{k+1} \in [0, 1 - \delta]$, $\alpha > 0$. For $n \geq k + p$ we define the sequence of rectangles $R_n$ by

$$R_n := \left[\alpha + \delta^{k+1}, \alpha + \delta^k (1 - \delta^p) - \sum_{l=k+p}^{n-1} \delta^l\right] \times [m, M].$$

(When $n = k + p$ we define the result of the empty summation to be 0.) We also suppose that $f$ is $n$-first class in $R_n$ for $n = k + p, k + p + 1, \ldots$. Then there are $x \in \left[\alpha + \delta^{k+1}, \alpha + \delta^k \cdot \left((1 - \delta^p) - \frac{\delta^p}{1 - \delta}\right)\right] \cap [0, 1 - \delta]$ and a sequence $x_j$ such that

(1) $x_j \in [x + \delta^j + 1, x + \delta^j]$ \quad ($j = 1, 2, \ldots$)

and

(2) $\sum_{j=2}^{\infty} |f(x_j) - f(x_{j-1})| < +\infty$.

**Proof.** By the choice of $p$,

$$I := \left[\alpha + \delta^{k+1}, \alpha + \delta^k \left((1 - \delta^p) - \frac{\delta^p}{1 - \delta}\right)\right] \neq \emptyset$$

and we let $x \in I \cap [0, 1]$. Since $f$ is $n$-first class in $R_n$ we have a function $y_n : [\alpha + \delta^{k+1}, \alpha + \delta^k (1 - \delta^p) - \sum_{l=k+p}^{n} \delta^l] \to \mathbb{R}$ and $x$ obviously belongs to the domain of $y_n$. We define the desired sequence by $x_j := x + \delta^j + 1$ for $0 < j < k + p$ and $x_j := y_j(x)$ for $j \geq k + p$, thus (1) is fulfilled. Since $f(y_j(x)) = (m + M)/2$ ($j \geq k + p$) obviously (2) is also fulfilled.

**Proof of the Theorem.** First we suppose that in each interval $[\alpha, \beta] \subset [0, 1 - \delta]$, $\sup\{f(x); x \in [\alpha, \beta]\} = +\infty$. Then letting $H_m := \{x \in [0, 1]; f(x) \leq m\}$ we have $\bigcup_{m=1}^{\infty} H_m = [0, 1]$ and hence, by Baire’s theorem, there is an $m_0$ and a subinterval $[\alpha_0, \beta_0]$ such that $H_{m_0}$ is dense in $[\alpha_0, \beta_0]$. Thus $f$, as a Darboux function, takes on the value $y = m_0 + 1$ in any subinterval of $[\alpha_0, \beta_0]$. Hence the statement of the theorem is obvious.

We can treat similarly the case when $\inf\{f(x); x \in [\alpha, \beta]\} = -\infty$ in any subinterval $[\alpha, \beta] \subset [0, 1 - \delta]$.
Therefore we can suppose that there is a rectangle $R_{-1} = [\alpha_0, \beta_0] \times [m_0, M_0]$ such that $f$ is in $R_{-1}$ and $\beta_0 < 1 - \delta$. Thus we can choose a number $j_0$ such that $\alpha_0 + \delta^{j_0} \leq \beta_0$. We put

$$R_{j_0} := [\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0}(1 - \delta^p)] \times [m_0, M_0]$$

and

$$\overline{R}_{j_0} := [\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0}] \times [m_0, M_0].$$

Obviously $f$ is in $R_{j_0}$ and $\overline{R}_{j_0}$. By our lemma we can suppose that the conditions of the lemma are not fulfilled by any $k, \alpha \in [0, 1 - \delta], \beta, m, M$. It follows that there is an index $j_1 \geq j_0 + p$ such that

$$\text{in } R_{j_1} := \left[\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0} \cdot (1 - \delta^p) - \sum_{l=j_0+p}^{j_1-1} \delta^l \right] \times [m_0, M_0]$$

$f$ is $j_1$-second class

and

for $j_0 + p \leq n < j_1$

$$R_n := \left[\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0} \cdot (1 - \delta^p) - \sum_{l=j_0+p}^{n-1} \delta^l \right] \times [m_0, M_0]$$

$f$ is $n$-first class.

From (3) it follows that there is

$$x' \in \left[\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0} \cdot (1 - \delta^p) - \sum_{l=j_0+p}^{j_1-1} \delta^l \right]$$

such that $f(x) \neq (m_0 + M_0)/2$ for $x \in [x' + \delta^{j_1+1}, x' + \delta^{j_1}]$. We put $a_1 := x'$.

From the Darboux property of $f$ it follows that we can choose $m_1$ and $M_1$ such that $M_1 - m_1 = \frac{1}{2}(M_0 - m_0)$ and $f$ is in $\overline{R}_{j_1} = [a_1 + \delta^{j_1+1}, a_1 + \delta^{j_1}] \times [m_1, M_1]$ and $\overline{R}_{j_1} \subset \overline{R}_{j_0}$. We put

$$R_{j_1} := [\alpha_1 + \delta^{j_1+1}, \alpha_1 + \delta^{j_1} \cdot (1 - \delta^p)] \times [m_1, M_1] \quad (\subset \overline{R}_{j_1} \subset \overline{R}_{j_0}).$$

Obviously

$$[\alpha_1 + \delta^{j_1+1}, \alpha_1 + \delta^{j_1}] \subset \left[\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0} \cdot (1 - \delta^p) - \sum_{l=j_0+p}^{j_1-1} \delta^l \right];$$

hence, by (4), $x$ belongs to the domain of $y_n$ ($j_0 + p \leq n < j_1$) for all $x \in [\alpha_1 + \delta^{j_1+1}, \alpha_1 + \delta^{j_1}]$.

Suppose we have defined $R_{j_n}$ for $n \leq i$. Then we repeat our process in $R_{j_i}$ and define the index $j_{i+1}$, the numbers $m_{i+1}, M_{i+1}$, the rectangles $R_{j_{i+1}}, \overline{R}_{j_{i+1}}$, and the functions $y_n$ ($j_i + p \leq n < j_{i+1}$). Thus, by induction we can define an infinite sequence of rectangles $R_{j_i}, \overline{R}_{j_i}$ ($i = 0, 1, \ldots$) of numbers $m_i, M_i$ ($i = 0, 1, \ldots$), and we also define the functions $y_n$ for $j_{i+1} > n \geq j_i + p, i \in \mathbb{N}$. Since $R_{j_{i+1}} \subset R_{j_i}$ ($i = 0, 1, \ldots$) and $R_{j_i}$ is a closed set, $\bigcap_{i=1}^{\infty} R_{j_i} \neq \emptyset$; let $(x, y) \in \bigcap_{i=1}^{\infty} R_{j_i}$. For
0 < n < j_0 + p we put \( x_n := x + \delta^{n+1} \). If there is an \( i \in \mathbb{N} \) such that \( j_i + p > n \geq j_i \), then we also put \( x_n := x + \delta^{n+1} \), otherwise we put \( x_n = y_n(x) \). Thus (a) of the theorem is obviously fulfilled. We also have that

\[
\sum_{n=jo+1}^{\infty} |f(x_n) - f(x_{n-1})| = \sum_{i=0}^{\infty} \sum_{n=j_i+p+1}^{j_i+1} |f(x_n) - f(x_{n-1})| + \sum_{i=1}^{\infty} \sum_{n=j_i}^{j_i+p} |f(x_n) - f(x_{n-1})| + \sum_{n=jo+1}^{j_0+p} |f(x_n) - f(x_{n-1})| =: A_1 + A_2 + A_3.
\]

Since \( f(x_n) = (M_i - m_i)/2 \) for \( j_i + p \leq n \leq j_{i+1} - 1 \), we have \( A_1 = 0 \). It is easy to check that \( (x_l; f(x_l)) \in \overline{R}_{j_l-1} \) for \( l \geq j_i - 1 \). Thus \( A_2 \leq \sum_{i=1}^{\infty} (p+1)(M_{i-1} - m_{i-1}) \). We also have that

\[ M_{i+1} - m_{i+1} = (M_i - m_i)/2 \quad (i = 0, 1, \ldots). \]

Thus \( A_2 < \infty \) and it completes the proof.

REFERENCES