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FOR EVERY CONTINUOUS f THERE IS AN ABSOLUTELY CONTINUOUS g SUCH THAT $[f = g]$ IS NOT BILATERALLY STRONGLY POROUS

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ABSTRACT. For any Darboux function $f\colon [0,1] \to \mathbf{R}$ and any $0 < \delta < 1$ there is a point $x \in [0, 1-\delta]$ and a sequence x_n such that

(a) $x_n \in [x + \delta^{n+1}, x + \delta^n]$ $(n = 1, 2, \dots)$ and

(b) $\sum_{n=2}^{\infty} |f(x_n) - f(x_{n-1})| < +\infty$.

Consequently, for every $f \in C[0,1]$ there is an absolutely continuous function g such that $\{x\colon f(x) = g(x)\}$ is not bilaterally strongly porous.

In [**1**] P. Humke and M. Laczkovich proved that every continuous function agrees with an absolutely continuous function on a set which is not bilaterally strongly $x^{1+\delta}$ porous (Theorem 3).

In fact, they proved that for any $f \in C[0,1]$ there is a point $x \in [0,1]$ and a sequence $x_n \to x$ such that

$$x_n \in [x + ((n+1)!)^{-1-\delta}, x + (n!)^{-1-\delta}]$$

and

$$|f(x_n) - f(x)| < n^{-1-\delta/2}$$

for $n > n_0$ and $\delta > 0$.

Let $g \in C[0,1]$ be linear on the intervals $[0,x]$, $[x_{n+1}, x_n]$ $(n = 1, 2, \dots)$, $[x_1, 1]$, and agree with f at the points x_n. Since

$$\sum_{n=n_0}^{\infty} |f(x_n) - f(x_{n-1})| < +\infty,$$

g is absolutely continuous and $f = g$ on a set which is not bilaterally strongly $x^{1+\delta}$-porous. We prove the following

THEOREM. *For any Darboux function $f\colon [0,1] \to \mathbf{R}$ and any $0 < \delta < 1$ there is a point $x \in [0, 1-\delta]$ and a sequence x_n such that*

(a) $x_n \in [x + \delta^{n+1}, x + \delta^n]$ $(n = 1, 2, \dots)$ *and*

(b) $\sum_{n=2}^{\infty} |f(x_n) - f(x_{n-1})| < +\infty$.

Defining the function g as above, we obtain the following

COROLLARY. *For every $f \in C[0,1]$ there is an absolutely continuous function g such that $\{x\colon f(x) = g(x)\}$ is not bilaterally strongly porous.*

DEFINITION 1. Let R be a rectangle $R = [\alpha, \beta] \times [m, M]$. We say that f is in R if graf $f|_{[\alpha,\beta]} \subset R$.

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From now on we suppose that we are given a positive number $\delta < 1$ and a Darboux function $f\colon [0,1] \to \mathbf{R}$.

DEFINITION 2. Suppose that we have a rectangle $R = [\alpha, \beta] \times [m, M]$ and a number $k \in \mathbf{N}$ such that $\delta^k < \beta - \alpha$ and f is in R. We say that f is k-first class in R if whenever $I \subseteq [\alpha + \delta^{k+1}, \beta]$ with $|I| = \delta^k(1-\delta)$, then there exists $x \in I$ such that $f(x) = (M+m)/2$. Thus we can define a function $y_k\colon [\alpha, \beta - \delta^k] \to \mathbf{R}$ such that for any $x \in [\alpha, \beta - \delta^k]$ we have $f(y_k(x)) = (m+M)/2$ and $y_k(x) \in [x + \delta^{k+1}, x + \delta^k]$. We say that f is k-second class in R if f is not k-first class in R; that is if there is an $x' \in [\alpha, \beta - \delta^k]$ such that $f(x) \neq (m+M)/2$ for $x \in [x' + \delta^{k+1}, x' + \delta^k]$.

We choose a number $p \in \mathbf{N}$ such that $(1 - \delta^p) - \sum_{n=p}^{\infty} \delta^n \geq \delta$.

LEMMA. *Suppose that $k \in \mathbf{N} \cup \{0\}$, $m < M \in \mathbf{R}$, $\alpha + \delta^{k+1} \in [0, 1-\delta]$, $\alpha > 0$. For $n \geq k+p$ we define the sequence of rectangles R_n by*

$$R_n := \left[\alpha + \delta^{k+1}, \alpha + \delta^k(1-\delta^p) - \sum_{l=k+p}^{n-1} \delta^l\right] \times [m, M].$$

(When $n = k+p$ we define the result of the empty summation to be 0.) We also suppose that f is n-first class in R_n for $n = k+p, k+p+1, \ldots$. Then there are

$$x \in \left[\alpha + \delta^{k+1}, \alpha + \delta^k \cdot \left((1-\delta^p) - \frac{\delta^p}{1-\delta}\right)\right] \cap [0, 1-\delta]$$

and a sequence x_j such that

$$x_j \in [x + \delta^{j+1}, x + \delta^j] \qquad (j = 1, 2, \ldots) \tag{1}$$

and

$$\sum_{j=2}^{\infty} |f(x_j) - f(x_{j-1})| < +\infty. \tag{2}$$

PROOF. By the choice of p,

$$I := \left[\alpha + \delta^{k+1}, \alpha + \delta^k \left((1-\delta^p) - \frac{\delta^p}{1-\delta}\right)\right] \neq \varnothing$$

and we let $x \in I \cap [0,1]$. Since f is n-first class in R_n we have a function $y_n\colon [\alpha + \delta^{k+1}, \alpha + \delta^k(1-\delta^p) - \sum_{l=k+p}^{n} \delta^l] \to \mathbf{R}$ and x obviously belongs to the domain of y_n. We define the desired sequence by $x_j := x + \delta^{j+1}$ for $0 < j < k+p$ and $x_j := y_j(x)$ for $j \geq k+p$, thus (1) is fulfilled. Since $f(y_j(x)) = (m+M)/2$ $(j \geq k+p)$ obviously (2) is also fulfilled.

PROOF OF THE THEOREM. First we suppose that in each interval $[\alpha, \beta] \subset [0, 1-\delta]$, $\sup\{f(x); x \in [\alpha, \beta]\} = +\infty$. Then letting $H_m := \{x \in [0,1]; f(x) \leq m\}$ we have $\bigcup_{m=1}^{\infty} H_m = [0,1]$ and hence, by Baire's theorem, there is an m_0 and a subinterval $[\alpha_0, \beta_0]$ such that H_{m_0} is dense in $[\alpha_0, \beta_0]$. Thus f, as a Darboux function, takes on the value $y = m_0 + 1$ in any subinterval of $[\alpha_0, \beta_0]$. Hence the statement of the theorem is obvious.

We can treat similarly the case when $\inf\{f(x); x \in [\alpha, \beta]\} = -\infty$ in any subinterval $[\alpha, \beta] \subset [0, 1-\delta]$.

Therefore we can suppose that there is a rectangle $R_{-1} = [\alpha_0, \beta_0] \times [m_0, M_0]$ such that f is in R_{-1} and $\beta_0 < 1 - \delta$. Thus we can choose a number j_0 such that $\alpha_0 + \delta^{j_0} \leq \beta_0$. We put

$$R_{j_0} := [\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0}(1 - \delta^p)] \times [m_0, M_0]$$

and

$$\overline{R}_{j_0} := [\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0}] \times [m_0, M_0].$$

Obviously f is in R_{j_0} and $\overline{R}_{j_0}$. By our lemma we can suppose that the conditions of the lemma are not fulfilled by any k, $\alpha \in [0, 1-\delta], \beta, m, M$. It follows that there is an index $j_1 \geq j_0 + p$ such that

$$\text{(3)} \qquad \text{in } R'_{j_1} := \left[\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0} \cdot (1 - \delta^p) - \sum_{l=j_0+p}^{j_1-1} \delta^l\right] \times [m_0, M_0]$$

f is j_1-second class

and

for $j_0 + p \leq n < j_1$ in

$$\text{(4)} \qquad R_n := \left[\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0}(1 - \delta^p) - \sum_{l=j_0+p}^{n-1} \delta^l\right] \times [m_0, M_0]$$

f is n-first class.

From (3) it follows that there is

$$x' \in \left[\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0}(1 - \delta^p) - \sum_{l=j_0+p}^{j_1} \delta^l\right]$$

such that $f(x) \neq (m_0 + M_0)/2$ for $x \in [x' + \delta^{j_1+1}, x' + \delta^{j_1}]$. We put $\alpha_1 := x'$.

From the Darboux property of f it follows that we can choose m_1 and M_1 such that $M_1 - m_1 = \frac{1}{2}(M_0 - m_0)$ and f is in $\overline{R}_{j_1} = [\alpha_1 + \delta^{j_1+1}, \alpha_1 + \delta^{j_1}] \times [m_1, M_1]$ and $\overline{R}_{j_1} \subset R_{j_0}$. We put

$$R_{j_1} := [\alpha_1 + \delta^{j_1+1}, \alpha_1 + \delta^{j_1}(1 - \delta^p)] \times [m_1, M_1] \qquad (\subset \overline{R}_{j_1} \subset R_{j_0}).$$

Obviously

$$[\alpha_1 + \delta^{j_1+1}, \alpha_1 + \delta^{j_1}] \subset \left[\alpha_0 + \delta^{j_0+1}, \alpha_0 + \delta^{j_0}(1 - \delta^p) - \sum_{l=j_0+p}^{j_1-1} \delta^l\right];$$

hence, by (4), x belongs to the domain of y_n ($j_0 + p \leq n < j_1$) for all $x \in [\alpha_1 + \delta^{j_1+1}, \alpha_1 + \delta^{j_1}]$.

Suppose we have defined R_{j_n} for $n \leq i$. Then we repeat our process in R_{j_i} and define the index j_{i+1}, the numbers m_{i+1}, M_{i+1}, the rectangles $R_{j_{i+1}}, \overline{R}_{j_{i+1}}$, and the functions y_n ($j_i + p \leq n < j_{i+1}$). Thus, by induction we can define an infinite sequence of rectangles R_{j_i}, $\overline{R}_{j_i}$ ($i = 0, 1, \dots$) of numbers m_i, M_i ($i = 0, 1, \dots$), and we also define the functions y_n for $j_{i+1} > n \geq j_i + p$, $i \in \mathbf{N}$. Since $R_{j_{i+1}} \subset R_{j_i}$ ($i = 0, 1, \dots$) and R_{j_i} is a closed set, $\bigcap_{i=1}^{\infty} R_{j_i} \neq \emptyset$; let $(x, y) \in \bigcap_{i=1}^{\infty} R_{j_i}$. For

$0 < n < j_0 + p$ we put $x_n := x + \delta^{n+1}$. If there is an $i \in \mathbf{N}$ such that $j_i + p > n \geq j_i$ then we also put $x_n := x + \delta^{n+1}$, otherwise we put $x_n = y_n(x)$. Thus (a) of the theorem is obviously fulfilled. We also have that

$$\begin{aligned}\sum_{n=j_0+1}^{\infty} |f(x_n) - f(x_{n-1})| &= \sum_{i=0}^{\infty} \sum_{n=j_i+p+1}^{j_{i+1}-1} |f(x_n) - f(x_{n-1})| \\ &\quad + \sum_{i=1}^{\infty} \sum_{n=j_i}^{j_i+p} |f(x_n) - f(x_{n-1})| \\ &\quad + \sum_{n=j_0+1}^{j_0+p} |f(x_n) - f(x_{n-1})| \\ &= : A_1 + A_2 + A_3.\end{aligned}$$

Since $f(x_n) = (M_i - m_i)/2$ for $j_i + p \leq n \leq j_{i+1} - 1$, we have $A_1 = 0$. It is easy to check that $(x_l; f(x_l)) \in \overline{R}_{j_{i-1}}$ for $l \geq j_i - 1$. Thus $A_2 \leq \sum_{i=1}^{\infty} (p+1)(M_{i-1} - m_{i-1})$. We also have that

$$M_{i+1} - m_{i+1} = (M_i - m_i)/2 \qquad (i = 0, 1, \dots).$$

Thus $A_2 < \infty$ and it completes the proof.

References

1. P. Humke and M. Laczkovich, *Typical continuous functions are virtually nonmonotone*, Proc. Amer. Math. Soc. **94** (1985), 244–248.

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