A DISTINGUISHING EXAMPLE IN \(k\)-SPACES

JOHN ISBELL

ABSTRACT. Not all locally compact spaces are \(k\)-spaces (that is, in the core-reflective hull of the compact Hausdorff spaces).

1. The largest known cartesian closed coreflective full subcategory of \(\text{Top}\) is the coreflective hull, say \(K_3\), of the category of spaces variously known as exponentiable, exponential, core-compact, or quasi-locally compact: the spaces whose topology is a continuous lattice. In \(K_3\) is \(K_2\), the coreflective hull of the locally compact spaces; and in \(K_2\) is the coreflective hull \(K_1\) of the compact Hausdorff spaces. It has been unknown whether \(K_1 = K_3\). This note distinguishes them; in fact, \(K_1 \neq K_2\). That answers Problem 5 of Herrlich [1].

Problem 6, whether \(K_2 = K_3\), remains. Also, the example is not sober, and thereby hangs another problem. Note that \(K_2\) and \(K_3\) certainly contain the same sober spaces, since every sober exponentiable space is locally compact [2].

If you want a \(T_1\) example, help yourself, for passage to the smallest containing \(T_1\) topology preserves local compactness and nonmembership in \(K_1\). (The example below is locally compact; anyway, the existence of examples implies the existence of locally compact examples, since “coreflective hull” is a closure operation.)

2. What prevents compact Hausdorff spaces (and even compact normal spaces) from generating \(K_2\) is this:

   If a compact normal space has subsets \(S_\alpha\) indexed by a non-\(\sigma\)-compact initial segment of the ordinals, the union of any proper initial segment of \(S_\alpha\)'s is closed, and the union \(U\) of all \(S_\alpha\) is open, then \(U\) is closed.

   PROOF. Suppose \(U\) is not closed. Let \(x_1\) be a point of some \(S_{\alpha_1}\). \(I_1 = \bigcup S_{\alpha_2}: \alpha \leq \alpha_1\) is closed and disjoint from the closed complement \(R\) of \(U\), so they have disjoint neighborhoods \(N_1, Q_1\). \(N_1 \neq U\), since \(U\), being nonclosed, meets \(Q_1\). Inductively, having \(x_1, \ldots, x_k, x_j\) in \(S_{\alpha_j}\), \(x_{j+1}\) outside a neighborhood \(N_j\) of \(I_j\), and \(N_k \neq U\) a neighborhood of \(I_1 \cup \cdots \cup I_k\), choose \(x_{k+1}\) in \(U - N_k\). \(I_{k+1}\) and \(R\) have disjoint neighborhoods, and the induction runs. But \(\bigcup I_j\) is a countable union, hence proper and closed; it is covered by the open sets \(N_j\), but by no finite number of them, which is absurd.

   Consider the space \(X\) consisting of the countable ordinals and \(\omega_1\), with closed sets the countable initial segments, \(\{\omega_1\}\), unions of two of those, and \(X\). Every subspace is compact; for if nonempty, it has a least element, a neighborhood of which contains the rest of the subspace except perhaps \(\omega_1\). So \(X\) is locally compact. However, for any continuous map \(f\) from a compact Hausdorff space to \(X\), by the...

Received by the editors January 27, 1986 and, in revised form, May 21, 1986.

1980 Mathematics Subject Classification. Primary 54D50.

Key words and phrases. \(k\)-space.
lemma above, $f^{-1}(\{\omega_1\})$ is open. Thus each such $f$ factors through $X^*$, which is $X$ with $\{\omega_1\}$ made open; and $X \in \mathcal{K}_2 - \mathcal{K}_1$.

3. Let $Y$ be a topological space and $Y_1$ the same set with the smallest containing $T_1$ topology, a subbase for which is given by the open sets of $Y$ and the complements of singletons. By Alexander’s Lemma, $Y_1$ is compact if $Y$ is. (It is a bit easier than Alexander’s Lemma.) Suppose $Y$ is locally compact, and consider a basic neighborhood $W = U - \{x_1, \ldots, x_n\}$ of $p \in Y_1$. For $i \leq n$, if $p \notin \{x_i\}^\circ$ in $Y$, $Y - \{x_i\}$ is a neighborhood of $p$; intersecting, we have $W = V - F$ where $V$ is a $y$-neighborhood of $p$ and $F$ is a (finite) set of points $x_i$ with $p \notin \{x_i\}^\circ$ in $Y$. Then $V$ contains a compact $(Y^*)$ neighborhood $N$ of $p$. $N - F$ is also compact in $Y$, for any open sets of $Y$ covering it cover $p$ and $F$. By the previous remark, $N - F$ is compact in $Y_1$. Finally, suppose $Y_1 \in \mathcal{K}_1$. The $\mathcal{K}_1$-coreflection $Y^*$ of $Y$ has a topology contained in the topology of $Y_1$ (since the continuous identity function $Y_1 \to Y$ factors through $Y^*$), and $p \in \{x\}^\circ$ in $Y$ is preserved in the coreflection because of the map $[0,1] \to Y$ taking $0$ to $p$ and the rest to $x$. Thus $Y^* \to Y$ is a homeomorphism, and $Y \in \mathcal{K}_1$.

REFERENCES


DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, BUFFALO, NEW YORK 14214