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CLOSED ORBITS OF AN ANOSOV FLOW AND THE FUNDAMENTAL GROUP

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ABSTRACT. We show that closed orbits of a transitive Anosov flow generate the fundamental group of the base manifold.

Introduction. Let φ_t: $M \to M$ be a transitive Anosov flow on a compact manifold. Fried [5] showed that $H_1(M, \mathbf{Z})$ is generated by homology classes of φ_t-closed orbits. In this note we show that homotopy classes of closed orbits generate the fundamental group $\pi_1(M)$. More precisely, for $\gamma \in \pi_1(M)$ we denote by $\langle \gamma \rangle$ the conjugacy class in $\pi_1(M)$ containing γ. Each closed orbit $\mathfrak{p}$ defines a conjugacy class $\langle \mathfrak{p} \rangle$. Our main result is

THEOREM. *If φ_t: $M \to M$ is a transitive Anosov flow, then those elements $\gamma \in \pi_1(M)$ with $\langle \gamma \rangle = \langle \mathfrak{p} \rangle$ for some φ_t-closed orbit $\mathfrak{p}$ generate $\pi_1(M)$.*

In some cases each conjugacy class contains an orbit cycle (i.e. a multiple of a closed orbit), hence the theorem trivially holds. But we have many examples of Anosov flows without this property; geodesic flows of surfaces and suspensions of Anosov diffeomorphisms.

We prove the theorem by an elementary argument on graphs. A Markov family for φ_t defines an oriented graph embedded in M such that each φ_t-closed orbit is approximated by an oriented closed path in this graph. In §1 we construct a covering of Markov families with the covering transformation group $\pi_1(M)$, and show that the associated graphs are connected. Next we show in §2 that the induced homomorphism of the fundamental group of the graph into $\pi_1(M)$ is surjective. Since the fundamental group of this graph is generated by free homotopy classes of oriented closed paths, we can get the assertion. In §3, using our main theorem, we improve Theorem B in [3] which is concerned with the analyticity of L-functions associated to φ_t.

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1. A covering of Markov families. Let π: $\tilde{M} \to M$ be the universal covering and $\tilde{\varphi}_t$: $\tilde{M} \to \tilde{M}$ the covering flow of φ_t. In this section we construct a covering of connected graphs associated to a Markov family for φ_t.

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An *oriented graph* (V, E) consists of a set of vertices V and a set of edges $E \subset V \times V$. We usually identify it with a 1-dimensional CW-complex equipped with an orientation on each 1-simplex. An *oriented path* $c = (v_0, \dots, v_n)$, $(v_i, v_{i+1}) \in E$, from v_0 to v_n is said to be *closed* if $v_0 = v_n$. We call (V, E) irreducible if for every pair $(v, w) \in V \times V$ there is an oriented path from v to w. An irreducible graph is connected as a CW-complex (the converse is, in general, not true).

We put

$$C = 1 + \sup\{ d\langle \tilde{\varphi}_t(p), p\rangle / t \mid 0 < t \leqslant 1,\ p \in \tilde{M} \}.$$

If $\varepsilon > 0$ is small enough, then for each $\tilde{x} \in \tilde{M}$ the $10\varepsilon C$-neighborhood of $\tilde{x}$ is contained in some covering sheet, and the stable and unstable sets $W_\varepsilon^{\mathrm{s}}(x)$, $W_\varepsilon^{\mathrm{u}}(x)$ $(x \in M)$ for φ_t play a role of coordinates (see [**8**, **9**]).

Let V be a Markov family of size α. (Here α is sufficiently small compared with $\varepsilon/10C$.) For the definition and the properties of a Markov family, see [**4**]. We use here the irreducible oriented graph (V, E) defined as follows. Since V is a finite family of disjoint φ_t-local cross sections, for each $x \in \Gamma(V) = \bigcup_{v \in V} v$ one can attach the smallest positive $t(x)$ $(\leqslant \alpha)$ with $\varphi_{t(x)}(x) \in \Gamma(V)$. We define E by

$$\{(v, w) \in V \times V \mid \text{there is } x \in v \cap \Gamma'(V) \text{ with } T(x) \in w\},$$

where $T\colon \Gamma(V) \to \Gamma(V)$ is the bijection given by $T(x) = \varphi_{t(x)}(x)$ and

$$\Gamma'(V) = \left\{ x \in \Gamma(V) \mid T^k(x) \in \bigcup_{v \in V} \operatorname{Int}(v) \text{ for every } k \in \mathbb{Z} \right\}$$

is a dense subset of $\Gamma(V)$. The transitivity of φ_t guarantees that (V, E) is irreducible.

We now construct an oriented graph associated to $\tilde{\varphi}_t$. Since the diameter of $v \in V$ is smaller than α, there exists $\tilde{v} \subset \tilde{M}$ such that

$$\pi^{-1}(v) = \bigcup_{\gamma \in \pi_1(M)} \gamma\tilde{v}, \qquad \gamma\tilde{v} \cap \tilde{v} = \varnothing \text{ if } \gamma \neq \mathrm{id}.$$

The family of $\tilde{\varphi}_t$-local cross sections $\tilde{V} = \{\gamma\tilde{v} \mid v \in V,\ \gamma \in \pi_1(M)\}$ similarly yields an oriented graph $(\tilde{V}, \tilde{E})$. Note that the bijection $\tilde{T}\colon \Gamma(\tilde{V}) \to \Gamma(\tilde{V})$ satisfies $\pi \circ \tilde{T} = T \circ \pi$ and $\Gamma'(\tilde{V}) = \pi^{-1}(\Gamma'(V))$. Therefore, if we define $P\colon \tilde{V} \to V$ by $P(\gamma\tilde{v}) = v$, then $P\colon (\tilde{V}, \tilde{E}) \to (V, E)$ is a covering of oriented graphs with the covering transformation group $\pi_1(M)$ (i.e. a covering of CW-complexes with compatible orientations). Although $\tilde{V}$ might be an infinite family, it plays a role of a Markov family for $\tilde{\varphi}_t$. One can find a strictly positive function $\tilde{f}$ and a bounded-to-one continuous map $\tilde{\Phi}$: $\Sigma(\tilde{V}, \tilde{E}, \tilde{f}) \to \tilde{M}$ such that

(i) $\tilde{\varphi}_t \circ \tilde{\Phi} = \tilde{\Phi} \circ \sigma(\tilde{f})_t$,

(ii) for each $\tilde{x} \in \Gamma(\tilde{V})$ there is $\xi \in \Sigma(\tilde{V}, \tilde{E})$ with $\tilde{\Phi}(\xi, 0) = \tilde{x}$.

Here $\Sigma(\tilde{V}, \tilde{E}, \tilde{f})$ denotes the suspension with $\tilde{f}$ of the subshift $\Sigma(\tilde{V}, \tilde{E}) = \{\xi = (\xi^i) \in \prod_{\mathbf{Z}} \tilde{V} \mid (\xi^i, \xi^{i+1}) \in \tilde{E}\}$ and $\sigma(\tilde{f})_t$ the suspension flow.

We devote the rest of this section to showing that $(\tilde{V}, \tilde{E})$ is connected. By the definition of a Markov family we may suppose that for each $\tilde{v} \in \tilde{V}$ there exists a $\tilde{\varphi}_t$-local cross section $\tilde{D}$ containing $\tilde{v}$ such that

(1) if $\tilde{v} \cap \tilde{\varphi}_{[-\alpha,\alpha]}(\tilde{v}') \neq \varnothing$, then $\tilde{v} \subset \tilde{\varphi}_{[-2\alpha,2\alpha]}(\tilde{D}')$.

(2) $\tilde{D} \times [-3\alpha, 3\alpha] \ni (p, t) \mapsto \tilde{\varphi}_t(p) \in \tilde{\varphi}_{[-3\alpha,3\alpha]}(\tilde{D})$ is a diffeomorphism.

LEMMA. *Let $\tilde{v}$, $\tilde{v}' \in \tilde{V}$. If there are points $\tilde{x} \in \tilde{v}$ and $0 < \tau \leqslant \alpha$ with $\tilde{\varphi}_\tau(\tilde{x}) \in \mathrm{Int}(\tilde{v}')$, then there is an oriented path from $\tilde{v}$ to $\tilde{v}'$.*

PROOF. Choose an open neighborhood U' of $\tilde{\varphi}_\tau(\tilde{x})$ so that $U' \cap \tilde{D}' \subset \mathrm{Int}(\tilde{v}')$, and put $U = \tilde{v} \cap \mathrm{Pr}(U' \cap \tilde{D}')$, where Pr: $\tilde{\varphi}_{[-3\alpha,3\alpha]}(\tilde{D}) \to \tilde{D}$ is given by $\tilde{\varphi}_t(\tilde{z}) \mapsto \tilde{z}$. Since $\Gamma'(\tilde{V})$ is dense in $\Gamma(\tilde{V})$, the set $U \cap \Gamma'(\tilde{V})$ is not empty. As $\Gamma'(\tilde{V})$ is $\tilde{T}$-invariant, we get the conclusion.

Now suppose the distance between $\tilde{v}$, $\tilde{v}' \in \tilde{V}$ is shorter than $8\alpha C$. Then we can conclude they are joined by a curve in the CW-complex $(\tilde{V}, \tilde{E})$ in the following manner. Put $v = P(\tilde{v})$ and $v' = P(\tilde{v}')$. Since φ_t-periodic points are dense in $\Gamma(V)$, there are φ_t-periodic points $x \in \mathrm{Int}(v)$ and $y \in \mathrm{Int}(v')$. The point $z = \langle x, y\rangle \in W_\varepsilon^s(\varphi_\mu(x)) \cap W_\varepsilon^u(y)$ $(|\mu| \leqslant \varepsilon)$ satisfies

$$d(z,x) \leqslant d\big(z, \varphi_\mu(x)\big) + d\big(\varphi_\mu(x), x\big) \leqslant 2\varepsilon C.$$

Let $\tilde{x} \in \tilde{v}$, $\tilde{y} \in \tilde{v}'$, $\tilde{z}$ be the points with $\pi(\tilde{x}) = x$, $\pi(\tilde{y}) = y$, $\pi(\tilde{z}) = z$, and $d(\tilde{x}, \tilde{z}) \leqslant 2\varepsilon C$, For some $0 \leqslant \nu \leqslant \alpha$, $\tilde{\varphi}_\nu(\tilde{z})$ is contained in $\Gamma(\tilde{V})$, and is denoted by $\tilde{\Phi}(\xi, 0)$ with some $\xi \in \Sigma(\tilde{V}, \tilde{E})$. Since $x \in \mathrm{Int}(v)$ is a φ_t-periodic point and $\lim_{t\to\infty} d(\tilde{\varphi}_t(\tilde{z}), \tilde{\varphi}_{t+\mu}(\tilde{x})) = 0$, we can find $t_0 > \alpha$, $\tilde{v}'' \in P^{-1}(v)$, and ω with $|\omega| \leqslant \alpha$ such that

$$\tilde{\varphi}_{t_0+\mu}(\tilde{x}),\ \tilde{\varphi}_{t_0+\omega}(\tilde{z}) \in \mathrm{Int}(\tilde{v}'').$$

Choose a positive integer n and positive τ $(\leqslant \alpha)$ so that $\tilde{\varphi}_\tau\langle\Phi(\sigma^n\xi, 0)\rangle = \tilde{\varphi}_{t_0+\omega}(\tilde{z})$, where σ denotes the shift operator. By the previous lemma, we have an oriented path $(\tilde{v}_0, \ldots, \tilde{v}_m)$ from ξ^n to $\tilde{v}''$. Hence ξ^0 and $\tilde{v}''$ are joined by an oriented path $(\xi^0, \ldots, \xi^n, \tilde{v}_1, \ldots, \tilde{v}_m)$. Similarly there is an oriented path joining $\tilde{v}$ and $\tilde{v}''$. Therefore we get a curve (not an oriented path) from $\tilde{v}$ to ξ^0 in $(\tilde{V}, \tilde{E})$. Applying the same argument to $\tilde{z}$ and $\tilde{y}$, we can conclude that $\tilde{v}$ and $\tilde{v}'$ are joined by a curve in $(\tilde{V}, \tilde{E})$.

Given arbitrary $\tilde{v}$, $\tilde{w} \in \tilde{V}$, we choose points $p_0, \ldots, p_s$ in $\tilde{M}$ so that $p_0 \in \tilde{v}$, $p_s \in \tilde{w}$, and $d(p_{j-1}, p_j) < 5\alpha C$. There are $\tilde{v}_j \in \tilde{V}$ and $0 \leqslant \tau_j \leqslant \alpha$ with $\tilde{\varphi}_{\tau_j}(p_j) \in \tilde{v}_j$, $j = 1, \ldots, s-1$. Since the distance between $\tilde{v}_{j-1}$ and $\tilde{v}_j$ is smaller than $8\alpha C$, $\tilde{v}$ $(= \tilde{v}_0)$ and $\tilde{w}$ $(= \tilde{v}_s)$ are joined by a curve in $(\tilde{V}, \tilde{E})$. Hence we obtain that $(\tilde{V}, \tilde{E})$ is connected.

2. Proof of theorem. We prove the theorem by using the covering of connected graphs $(\tilde{V}, \tilde{E}) \to (V, E)$. Since (V, E) is irreducible, there exist oriented closed paths $c_1, \ldots, c_m$ whose homotopy classes generate the fundamental group $\pi_1(V, E)$ of the CW-complex (see [**2**, Lemma 1-1]). Here we recall how (V, E) approximates the flow φ_t. Choose a point $x(v) \in v$ for each $v \in V$, and define a continuous map ι: $(V, E) \to M$ by $\iota(v) = x(v)$ and $\iota(v, w) =$ (the minimal geodesic joining $x(v)$ and $x(w)$). By Bowen's result [**4**] we get that for each oriented closed path c the loop $\iota(c)$ is free homotopic to some φ_t-orbit cycle (see [**2**] for details). Therefore there are φ_t-closed orbits $\mathfrak{p}_j$ such that $\iota(c_j)$ is free homotopic to $\mathfrak{p}_j^{k_j}$ for some positive integer k_j. Hence the free homotopy classes of $\mathfrak{p}_j$ generate $\iota_*(\pi_1(V, E))$. What we have to do is to check that ι_* is surjective. Let $\tilde{v} \in \tilde{V}$. The connectedness of $(\tilde{V}, \tilde{E})$

guarantees that for each $\gamma \in \pi_1(M)$ there is a curve l from $\tilde{v}$ to $\gamma\tilde{v}$ for each $\gamma \in \pi_1(M)$. Since $\tilde{v}$ is contained in a covering sheet, the homotopy class of $\iota(P(l))$ coincides with γ. Hence ι_* is surjective and we get the theorem.

3. A remark on the analyticity of L-functions. It has been noted that the zeta function of an Anosov flow φ_t: $M \to M$ is useful in the study of φ_t-closed orbits (for example see [**3**, **5**, **7**, and **10**]). As an analogue of Selberg's zeta functions, we [**3**] defined the L-function for a flow ψ_t: $X \to X$ associated to an N-dimensional unitary representation ρ: $\pi_1(X) \to U(N)$ by

$$L(s;\rho,\psi_t) = \prod_{\mathfrak{p}} \det\{I - \rho(\langle \mathfrak{p}\rangle)\exp\langle -s\tau(\mathfrak{p})\rangle\}^{-1},$$

where $\mathfrak{p}$ runs over all ψ_t-closed orbits and $\tau(\mathfrak{p})$ denotes the minimal period of $\mathfrak{p}$.

Let $h(\varphi_t)$ denote the topological entropy of φ_t. Combining Theorem D in [**3**] with our theorem, we can conclude

PROPOSITION. *Let φ_t: $M \to M$ be a transitive Anosov flow. If ρ: $\pi(M) \to U(N)$ is irreducible and $N \geqslant 2$, then the L-function $L(s;\rho,\varphi_t)$ is holomorphic in some neighborhood of $\mathscr{R}e\, s \geqslant h(\varphi_t)$.*

This result improves Theorem B in [**3**]. Actually by Manning-Bowen's counting lemma [**4**], the problem of the analyticity of $L(s;\rho,\varphi_t)$ was reduced to that of $L\langle s;\rho\circ\iota_*,\sigma(f)_t\rangle$; here $\sigma(f)_t$ denotes the suspension flow on $\Sigma(V,E,f)$. By Theorem D in [**3**], if $N \geqslant 2$ and $\rho\circ\iota_*$ is irreducible then $L\langle s;\rho\circ\iota_*,\sigma(f)_t\rangle$ is holomorphic in some neighborhood of $\mathscr{R}e\, s \geqslant h\langle\sigma(f)_t\rangle = h(\varphi_t)$. Since ι_*: $\pi_1(V,E) \to \pi_1(M)$ is surjective, $\rho\circ\iota_*$ is irreducible if and only if ρ is irreducible, hence we get the assertion.

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