ON THE PURELY INSEPARABLE CLOSURE OF RINGS

SHIZUKA SATO

ABSTRACT. Let $K \subseteq R$ be commutative rings with identity 1. Let $D = \{D_i\}$ be a higher derivation of $R$. We shall prove in this paper that if $K$ is invariant with respect to $D$, the purely inseparable closure $K_R$ of $K$ in $R$ is invariant with respect to $D$ and the formal power series ring $K_R[[t]]$ is purely inseparably closed in $R[[t]]$.

1. Introduction. Let $K \subseteq R$ be commutative rings with identity 1. Let $D = \{D_i\}$ be a higher derivation of $R$. We shall write $D(K) \subseteq K$ if $D_i(K) \subseteq K$ for all $i$. We shall call $R$ purely inseparable over $K$ if for every element $x$ of $R$ there is an integer $n$ such that $(x \otimes 1 - 1 \otimes x)^n = 0$ in $R \otimes_K R$ (cf. [1]). It is immediately proved that there exists the maximum purely inseparable subalgebra over $K$ contained in $R$. We shall call this subalgebra the purely inseparable closure of $K$ in $R$ and we shall write $K_R$. We shall prove that $D(K) \subseteq K$ implies $D(K_R) \subseteq K_R$ and the formal power series ring $K_R[[t]]$ is purely inseparably closed in $R[[t]]$.

2. The purely inseparable closure. Let $K \subseteq R$ be commutative rings with identity 1. Let $\{K_\lambda\}_{\lambda \in \Lambda}$ be the set of purely inseparable subalgebras over $K$ contained in $R$ and let $K_R$ be the subalgebra of $R$ generated by $\{K_\lambda\}_{\lambda \in \Lambda}$ over $K$. Then it is evident that $K_R$ is the maximum purely inseparable subalgebra over $K$ in $R$. We shall call $K_R$ the purely inseparable closure of $K$ in $R$. If $K_R = K$, we shall call $K$ purely inseparably closed in $R$.

EXAMPLE. Let $K$ be a field of characteristic $p > 0$ and let $X$, $Y$, and $Z$ be three indeterminates over $K$. Let $R = K[X,Y,Z]/(Y^p - X) = K[x,y,z]$. Then $R$ is a domain and $K[x,y]$ is purely inseparable over $K[x]$. Assume $S$ is a purely inseparable subalgebra over $K[x]$ containing $K[x,y]$. An element $f \in S$ can be written in the form

$$f = \sum_{i=0}^{n} g_i(x,y)z^i, \quad g_i(x,y) \in K[x,y].$$

Then we have $f^{p^m} \otimes 1 - 1 \otimes f^{p^m} = 0$ for some integer $m$ in $S \otimes_K S$. Since $g_i(x,y)^{p^m} \in K[z]$, we have

$$\sum_{i=1}^{n} g_i(x,y)^{p^m} (z^{ip^m} \otimes 1 - 1 \otimes z^{ip^m}) = 0$$

and hence

$$g_i(x,y)^{p^m} = 0 \quad \text{for all } i \geq 1.$$
Since $R$ is a domain $D$, we have $g_i(x, y) = 0$ for all $i$ and $f = g_0(x, y)$. Therefore we have $S \subseteq K[x, y]$ and it follows that the purely inseparable closure $\overline{K[x]}_R$ of $K[x]$ in $R$ is $K[x, y]$.

A higher derivation $D$ of $R$ is an infinite sequence $D = \{D_0, D_1, D_2, \ldots\}$ of mappings $D_i$ of $R$ into $R$ such that

$$
D_i(x + y) = D_i(x) + D_i(y),
$$

$$
D_i(xy) = \sum_{j=0}^{i} D_j(x)D_{i-j}(y) \quad \text{for } x, y \in R,
$$

$$
D_0(1) = 1.
$$

Let $D = \{D_i\}$ be a higher derivation of $R$ and let $e^{tD} = D_0 + tD_1 + t^2D_2 + \cdots$ ($t$ is a variable). Then $e^{tD}$ is a homomorphism of $R$ into $R[[t]]$.

**Lemma 1.** Let $K \subseteq S \subseteq R$ be commutative rings with identity 1. Let $D = \{D_i\}$ be a higher derivation of $R$. Assume $S$ is a subalgebra of $R$ purely inseparable over $K$. If $D(K) \subseteq K$, then $e^{tD}(S)$ is a purely inseparable subalgebra of $R[[t]]$ over $e^{tD}(K)$.

**Proof.** Let $L$ be a reduced $e^{tD}(K)$-algebra and let $\psi, \phi$ be $e^{tD}(K)$-algebra homomorphisms of $e^{tD}(S)$ into $L$ such that $\psi i = \phi i$ where $i$ is the canonical injection of $e^{tD}(K)$ into $e^{tD}(S)$. We shall consider the commutative diagram:

$$
\begin{array}{c}
R \\
\uparrow \\
S \\
\uparrow i \\
K \\
\uparrow i \\
\end{array} \longrightarrow \begin{array}{c}
R[[t]] \\
\uparrow \\
e^{tD} \\
\uparrow i \\
e^{tD}(K) \\
\uparrow i \\
0
\end{array} 
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array} 
\begin{array}{c}
L \\
\phi \\
i \\
i \\
0
\end{array} 
\begin{array}{c}
\psi \\
\phi \\
i \\
i \\
0
\end{array} 
$$

Then, since $L$ is regarded as a $K$-algebra by $e^{tD}$, we have $\psi ie^{tD} = \phi ie^{tD}$. Hence we have $\psi ie^{tD} = \psi e^{tD}i$ and $\phi ie^{tD} = \phi e^{tD}i$. Since $S$ is purely inseparable over $K$, it holds that $\psi e^{tD} = \phi e^{tD}$ and hence $\psi = \phi$. Therefore $e^{tD}(S)$ is purely inseparable over $e^{tD}(K)$.

**Lemma 2.** Using the same terminology as in Lemma 1, let $T$ be an intermediate ring as $K \subseteq T \subseteq R$. Then if $S$ is a subalgebra of $R$ purely inseparable over $K$ then $ST$ is purely inseparable over $T$.

**Proof.** Let $U$ be a reduced $T$-algebra and let $\psi, \phi$ be $T$-algebra homomorphisms of $ST$ into $U$ satisfying $\psi i = \phi i$. We shall consider the commutative diagram:

$$
\begin{array}{c}
0 \\
\uparrow i \\
S \\
j \uparrow i \\
ST \\
\uparrow i \\
U \\
\end{array} \longrightarrow \begin{array}{c}
0 \\
\uparrow i \\
K \\
j \uparrow i \\
T
\end{array} 
\begin{array}{c}
\longrightarrow \\
\psi \\
\phi \\
i \\
i
\end{array} 
\begin{array}{c}
0 \\
\uparrow i \\
0
\end{array} 
$$

$(i, j$ are the canonical injections). From $\psi i = \phi i$ we have $\psi ji = \phi ji$ and hence $\psi ji = \phi ji$. Since $S$ is purely inseparable over $K$, it holds that $\psi = \phi$ and therefore $ST$ is purely inseparable over $T$. 

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LEMMA 3. Let $K \subseteq R$ be commutative rings with identity 1 and let $f$ be a mapping of the product space $R[[t]] \times R[[t]]$ into $(R \otimes_K R)[[t]]$ ($t$ is a variable) such that
\[
f\left(\sum a_i t^i, \sum b_i t^i\right) = \sum_{k=0}^\infty \sum_{i+j=k} (a_i \otimes b_j) t^k, \quad a_i, b_j \in R.
\]
Then $f$ is bilinear and hence we have a linear mapping $\bar{f}$ of $R[[t]] \otimes_K R[[t]]$ into $(R \otimes_K R)[[t]]$. By a simple calculation, this mapping is a ring homomorphism.

THEOREM 1. Let $K \subseteq R$ be commutative rings with identity 1 and let $D = \{D_i\}$ be a higher derivation of $R$. If $D(K) \subseteq K$, we have $D(KR) \subseteq KR$.

PROOF. Let $S$ be the subalgebra of $R$ generated by $\{D_i(a)\}_{a \in KR,t}$ over $KR$. By Lemmas 1 and 2, $e^{tD}(KR)K[[t]]$ is purely inseparable over $e^{tD}(K)K[[t]] = K[[t]]$. Therefore, for any element $a \in KR$, there is an integer $n$ such that
\[
ed^{tD}(a) \otimes 1 - 1 \otimes ed^{tD}(a)^n = 0
\]
in
\[
e^{tD}(KR)K[[t]] \otimes K[[t]] e^{tD}(KR)K[[t]].
\]
Since $e^{tD}(KR)K[[t]] \subseteq S[[t]]$, we have $e^{tD}(a) \otimes 1 - 1 \otimes e^{tD}(a)^n = 0$ in $S[[t]] \otimes_K S[[t]]$ and hence
\[
ed(e^{tD}(a) \otimes 1 - 1 \otimes e^{tD}(a))^n = 0
\]
in $(S \otimes_K S)[[t]]$ by Lemma 3. By a simple calculation we have
\[
\left[e^{tD}(a) \otimes 1 - 1 \otimes e^{tD}(a) - \sum_{i=0}^{m-1} (D_i(a) \otimes 1 - 1 \otimes D_i(a)) t^i\right]^{n2^n} = 0
\]
in $(S \otimes_K S)[[t]]$ and hence $(D_m(a) \otimes 1 - 1 \otimes D_m(a))^{n2^m} = 0$ in $S \otimes_K S$. Since $S$ is generated by $\{D_i(a)\}_{0 \leq i < \infty, a \in KR}$ over $KR$, $S$ is purely inseparable over $K$. Therefore we have $KR = S$ and $D(KR) \subseteq KR$.

THEOREM 2. Let $K \subseteq R$ be commutative rings with identity 1. Then $KR[[t]]$ is purely inseparably closed in $R[[t]]$.

PROOF. Let $D = \{D_k\}$ be a sequence of mappings of $R[[t]]$ into $R[[t]]$ such that $D_k(\sum a_i t^i) = a_k t^k$ for $\sum a_i t^i \in R[[t]], \quad k = 0, 1, 2, 3, \ldots$. Then $D$ is a higher derivation of $R[[t]]$ such that $D(KR[[t]]) \subseteq KR[[t]]$.

Let $T$ be the purely inseparable closure of $KR[[t]]$ in $R[[t]]$ and let $\alpha = \sum a_i t^i \in T$. By Theorem 1 we have $D_k(\alpha) \in T$ and hence $(\alpha \otimes 1 - 1 \otimes \alpha) t^n = 0$ in $T \otimes_K T$ for some integer $n$. Let $S$ be a subring of $R$ generated by coefficients of elements of $T$. Then we have $(\alpha \otimes 1 - 1 \otimes \alpha)^n (t^n \otimes 1) = 0$ in $S[[t]] \otimes_K S[[t]]$ and hence by Lemma 3 $(\alpha \otimes 1 - 1 \otimes \alpha)^n = 0$ in $S \otimes_K S$. By the definition of $S$, $S$ is purely inseparable over $KR$. Since $KR$ is purely inseparably closed in $R$, we have $KR = S$. Therefore it follows $T = KR[[t]]$ and hence $KR[[t]]$ is purely inseparably closed in $R[[t]]$. 

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FACULTY OF ENGINEERING, OITA UNIVERSITY, OITA 870-11, JAPAN